[SQUEAKING]

[RUSTLING]

[CLICKING]

HONG LIU: Good. Yeah. So last lecture, we discussed Noether's theorem. For every symmetry, for every continuous symmetry, there's a conserved current. And then we also started talking about relativistic quantum mechanics. How we want to unify special relativity in quantum mechanics.

So the most immediate idea for that is what's called the relativistic quantum mechanics. And the most immediate generalization of the Schrodinger equation-- so if you have-- so at the end of last lecture, we talked about, say, the most immediate generalization of the Schrodinger equation, which-- so if I have E equal to, say, p squared divided by 2m, and then you go to non-relativistic quantum mechanics Schrodinger equation.

And now if you have E squared equal to p squared plus m squared for relativistic particle, and then you get what's called the Klein-Gordon equation. And again, this side has the implication of the wave function. So this describe-- and then this-- so as a generalization of this, then this means to describe the quantum mechanics of relativistic free particle, say, of mass m, of mass m.

So here, the psi pi t, x is the wave function of a relativistic particle. Relativistic particle of mass m. And we also notice that this equation actually is the same as the simplest field theory equation. So we also talked about a simplest scalar field theory. Classical. So here is a simple-- a classic simplest scalar field theory.

So this theory you can write down the action of the form. So this is the simplest theory you can write down. And then relativistic environment theory. And then the equation of motion of this-- so this is-- you view this as a classical field. And again, this has the equation of motion have exactly the same form as this equation.

But now, here, phi, again, is a function of t, x. Now has a completely different physical interpretation. So here is-- this is a classical field. OK. So this is a classical field. OK. So in this case, the interpretation of the x in here, and in here, it's very different.

So not only phi and the psi, the physical interpretation are different. The physical interpretation of x also here are different. Here, x is just a label. Is a label for the location in the space which we define this field. But here, the x is the eigenvalue of the position operator for this relativistic particle. And so they have very different physical interpretation.

And so let me just label this equation by 1 and label this by 2 and this by 2 prime. So we also mentioned that this one has-- the interpretation of this as the wave function for relativistic-- yeah, for-- yeah, so the equation for relativistic quantum mechanics has a number of difficulties.

So the first is that, as you will show in your pset 2, there's no sensible-- no sensible way to define a positive definite probability density. So if you want to interpret this as a wave equation, then you must have a wave-- then you must have a probability density, because in quantum mechanics, probability should be conserved.

And the second difficulty is that there's a negative energy state because of the square. Because when you take the square root, then you get the minus sign, and then there's a negative energy state, which you cannot avoid in quantum mechanics even though classically you can just throw them away by hand.

And the third thing we mentioned at the end, we said for a relativistic wave equation, you can describe a fixed number of particles. So the particle number cannot change. So this equation describe a single particle. And if you want to describe two particles, then you need to write down a separate equation for a different wave function. So this is for the two-particle wave function will be like this and et cetera. OK.

But this does not really make sense in the relativistic system because we know that in the relativistic system, E equal to mc square, in any case, you have enough energy, then you should be able to create particles. And then that means the number of particles in the given process is not conserved.

So if you want to use your quantum mechanics to describe a process, and then that-- you cannot have a formalism, which the number of particles is fixed, which you cannot change. And so this is actually the most fundamental difficulty, is that you cannot change the number of particles.

And related to this difficulty is this interpretation. Here, we say-- now if you want to-- we're saying here, there's a fundamental asymmetry between the t and x. Also, yeah, maybe let me put it as 4, there's also fundamental-- an additional difficulty, there's a fundamental asymmetry between t and x.

So here, in the wave equation, t is just a parameter, which will be describing the evolution, but the x is the eigenvalue of quantum operators-- eigenvalues of quantum operators. Say, corresponding to, say, hat. By putting a hat, we denote the corresponding a quantum operator. Yeah, so this is the eigenvalue of position operators.

And this asymmetry become even more pronounced, say, if you can see the two particles. You have two x's here, but the only one t. OK.

So those-- because of those fundamental difficulties-- so if you connect this i to iv, so we conclude that the relativistic quantum mechanics defined in the sense that you write down a wave equation, and for wave function, don't even-- it does not-- that's not-- cannot be the fundamental description. But yeah, but relativistic quantum mechanic just refers to this kind of wave equation.

So at most, this can be approximated approximation at most. This is approximate description in situations, say, there's no particle creation or annihilation. So in case which your particle number is fixed, and then you can use this as approximation, but it cannot be a fundamental description.

For example, later we will talk about the fermionic version of this wave equation. So this will describe a particle without spin. So later we will describe the analogous equation for electrons for spin half, and then that can indeed be used to describe the electron in a hydrogen atom as far as you don't create new electrons, et cetera.

Anyway, so relativistic quantum mechanics can only be described as some kind of-- considered as an approximate description. But now if you want to unify special relativity and quantum mechanics together, it turns out that the right formulation is just quantum field theory. So it turns out that the quantum field theory addresses these difficulties. OK.

So it turns out that the right way-- so if we want to describe quantum mechanics, say, of relativistic particles of mass m-- so we want to do here, it turns out, the proper thing to do, which is a little bit unintuitive at first sight, is to start with this field theory, which seemingly have nothing to do with relativistic particles, but to start with this classical field theory and then quantize it.

It turns out, once you treat this theory as a quantum field theory, and this becomes a theory of arbitrary number of relativistic particles of mass m. And so that's the non-intuitive part, and that's one of the miracle, say, of the field theory, is that automatically gives you a formalism for treating arbitrary number of particles. And yeah.

And also, in field theory-- so both t and x are parameters even though x only labels your location. So both t and x are parameters, and so you can easily to make them to be on equal ground to be compatible with special relativity. OK. Good? So any questions on this? OK. So we will see that the right framework is quantum field theory. OK.

So finally, as a last motivation for quantum field theory, we quickly describe the last-- so the field theory can also arise at the limit of discrete systems. And this is the most relevant for condensed matter physics. For example--

So let's just consider, say, some-- yeah, let's consider 8.03 example. So let's imagine you have this number of particles, a number of the atoms, say, on the chain. And then they're connected by some springs between them. So this is the-- yeah, consider this to be infinite. And the spacing between them, say, is a.

So atom, I fixed on some lattice points. And the lattice spacing is a. So yeah, so we can label the-- say, each particle by their position. For example, this is x0, this is x1, this is x2, et cetera. OK. And and the typical particle is xn. At the location of a n-th particle is xn.

And so we can also introduce the deviation between the equilibrium position of each particle. So let's call it eta n. So now let's consider the dynamics of eta n for this theory. And so this is just a deviation of the n-th particle from its equilibrium position. So eta n 0 is equilibrium position.

So now so now if you write down the Lagrangian for this system, so we can easily do, you just write T minus V. So T is the kinetic energy and V is the potential energy. So we can just write it as sum over n over all particles. And then let's assume they have the same mass, let's write mu eta n dot square. So this is a kinetic term, so the mu is the mass for each particle.

And then their potential-- yeah, let's assume at each point, there is also-- yeah, let's just-- yeah, then there's a interaction because each particle are connected by the spring, and so they are a harmonic force between neighboring particles. And now let's imagine also there's a harmonic potential which trap this particle itself at each location. So this is a very simple spring and the particle problem which you encounter, say, in 8.03. Is this problem clear?

OK. I assume most of you have seen this problem before. And your task in 8.03 is actually to find the local modes, say, of this system. And in 8.03, you also described that we can-- and a go to zero limit.

So if the lattice spacing is very small. And if we are only interested in the behavior of the system at a very large distance, say the distance is much larger than a equals to-- much larger than a, then you can essentially treat this system as a continuum. You don't have to resolve individual particles.

And so we can just-- equal to newer limit. So it can be true to the chain as a continuum of particles. And so each eta n t, you replace it by eta x, t. So x label is position. So x label is position, t describe the dynamics. So eta is the deviation at the location x, and it's depend on t. So this is the oscillator.

And then sum over n in the Lagrangian, then we can replace it by integral over dx. And now you just treat this as a one-dimensional continuum. So integration over dx. But of course, here, there's a label-- a lattice spacing, so the so the infinitesimal here, the element is a. The a times the sum over n, you can replace it by dx. That is the lattice spacing.

And now you can just write this Lagrangian in terms of continuum theory. Now you can write this Lagrangian in terms of continuum theory, and then let's just do it. So we can write it-- yeah, let me just write one more step. So you can write it as sum over a. We take the a factor out because the a factor out has to be changing to integration.

And then you have 1/2 mu divided by a eta n squared minus 1/2 lambda a. So I just slightly rewrite this Lagrangian so that it's easy to take the continuum limit. So we have taken the factor of a out, but for this term, because this concerns the difference between the two, and we also divided by a in the downstairs, and then we need to multiply a upstairs, and then there's a in the front.

And now the continuum limit, you can just replace this by an integral. And now I just-- can just write it as 1/2 mu tilde eta dot squared. So now eta n just replaced by eta x, t. And here, let me call it lambda tilde. Partial x eta square. And this term, you can just replace it by the derivative of eta. And then this has just become 1/2 sigma tilde eta square.

And the mu tilde, of course, is mu divided by a. Lambda tilde is lambda times a. And sigma tilde, this is sigma divided by a. So the continuum limit is that this quantity has to be fixed. The tilde, the quantity has to be fixed. And then we have a continuum Lagrangian.

And then we have a classical field theory. And this theory is essentially the same as that theory. So if you take this factor mu tilde out, if we take this vector mu tilde out-- so let me just take this factor of mu tilde out in the front, just up to a wall factor. And here is lambda tilde divided by mu tilde. We call it-- let's call it v square.

And this becomes sigma tilde divided by mu tilde, let's call it m squared. So the V squared is equal to mu tilde is equal to the lambda tilde divided by mu tilde, and m squared is equal to sigma tilde squared by mu tilde. And then this is just essentially identical to that theory when V equal to 1.

So when V equal to 1, become the same, is just 2, the equation 2. So we could do, of course, corresponding to relativistic case, speed of light, but in general, this describe-- can-- yeah. This can describe-- but in general, this is a non-relativistic-- in general, this can be just a long relativistic field theory for other values of V.

So this is-- so even though this example is very simple, but this is actually a very general way that we can treat many condensed matter systems, which often involve a lattice, say, because the solid, you can imagine all the atoms are on the lattice, et cetera.

And if you're only interested in the very macroscopic behavior, then you can treat solid as a continuum. And now you can now if you're interested in the mechanics of such a system, then the quantum field theory then naturally arises. OK. Good. Any questions on this example? Yes?

STUDENT: So that, it goes to 1 is the same as lambda and mu being comparable? So what does that, I guess, physically mean?

HONG LIU: Sorry?

STUDENT: So the limit--

HONG LIU: Yeah, yeah, yeah, yeah.

STUDENT: Like, the same lambda as the same as mu?

HONG LIU: Yeah, yeah.

STUDENT: So what is that physically mean? Like, the strength of--

HONG LIU: Yeah, yeah. This goes right into the case that the-- yeah, it just tells you that the relativistic limit is special. Happens at very special point.

STUDENT: But I guess, why is that the relativistic-- To me, lambda is like the strength of your spring, and then--

HONG LIU: Yeah.

STUDENT: --use your mass.

HONG LIU: Right.

STUDENT: Without that being comparable, how does that--

HONG LIU: Yeah.

STUDENT: Because it's not the same thing as the relativistic--

HONG LIU: Yeah. There's not much you can read from here. Yeah, yeah. It's just like when you choose some special parameters, then you can have a relativistic limit, yeah. Other questions? Yes?

STUDENT: Yeah, so you said that you could use this to treat some condensed matter problems?

HONG LIU: Yeah.

STUDENT: Yeah, so these are all scalars, right?

HONG LIU: You can also have-- you can also-- you mean-- you can also have tensors or vectors, yeah.

STUDENT: OK.

HONG LIU: Yeah.

STUDENT: So like what would you treat with this? Like phonons.

HONG LIU: Oh, you can treat-- yeah. For example, you can treat phonons. You can also treat spins and say, for example, if you have an Ising model, just consider the lattice of spins, and then the average spin, and then you can treat it as a scalar field, and then again, you can write down the field theory. Yeah.

And actually, the breakthrough of the phase transition in condensed matter physics to understand what phase transition is really about and describe the behavior of the phase transition, and precisely coincided with the development of field theory. And yeah, actually, increased our understanding of quantum field theory, yeah. Good. Other questions?

OK, good. Just to summarize what we have discussed so far, all paths leads to QFT. So we have described three paths, but they are pretty general. First is that the quantum dynamics-- we often interested in quantum dynamics of some classical fields, say, such as, say, electric magnetic field or spacetime metric if you are interested in gravity, et cetera.

So in this case, we already have the classical field theory, but we know the world is quantum and we want to understand what's the quantum version of it. And the second is that it unifies special relativity plus quantum mechanics. So you need the field theory to unify them. And the third way is that it's the large distance description of these grid systems. OK.

So yeah, just combining all three elements together, they cover many, many areas of physics. They cover many, many areas of physics. Good. So now we can say a little bit about the plan for the whole semester. So here is the plan. So this is like just rephrase of the outline, which--

So the first thing we do in chapter 2-- so here is chapter 1. Chapter 2, we discussed the simplest field theory just to squash this equation 2. The theory of 2. 2 and 2 prime. Prime is its equation of motion. So yeah, we-- in physics, we always start with the simplest example. We always start with the simplest example. And so that is the one we will start with.

So what we will see is that this describes-- that field theory describe spin is, is both free massive particles. OK, so we will see, when, we quantize that theory 2, and then we get the theory of free massless-- free spinless massive particles. OK.

So you say, oh, that's a little bit boring because in this series, free-- the particle-- but free means they don't interact. The particle, they just don't interact. And then in chapter 3, we will add interactions. We will describe how to treat interactions. So we going to introduce interactions and tell you how to treat the interactions between those particles.

Then in chapter 4, we go to the real physics. So the scalar fields is also real. Say, for example, it can be used to describe the Higgs boson. But the Higgs boson maybe is a little bit far from what we normally think about.

So in chapter 4, we will go to something which is much closer. We'll talk about the theory of electron. So this is called the Dirac theory. So this theory describes free spin half particles. So this is a theory of electrons. OK. When we neglect these interactions. So this is the free spin half particles.

And then we move on to the Maxwell theory. So this is the theory of the quantum electric and magnetic field. So when you quantize the Maxwell theory, say, without source, the vacuum Maxwell theory, and you find you get free-- again, there's no interaction-- massless spin 1 particle. The theory of massless spin 1 particle.

So this is what we call the photon. So this is a quantum for electromagnetic fields. And then-- sorry. Did I-- so this should be chapter 5 now. I think I lost my count. So now go to chapter 6, we combine the 4 and the five together, combine electrons.

So photon normally we denote by gamma. Combine the theory of electron and the photon together, and then plot the interactions between them. And then we get the so-called quantum electrodynamics. So this is called QED.

So QED is very general. It essentially covers all the quantum phenomena-- yeah, a macroscopic phenomenon up to, say, weak interactions and strong interactions. If you don't go inside the nucleus or don't go to very high energy, I think that covers essentially most of the physics.

And yeah. And then we will describe how to-- yeah, and then that will be the end of this course. So do you have any questions on this? OK. So this is a road map. Yes?

STUDENT: Do these chapters correspond off of those chapters in the textbook or just chapters in the lecture notes?

HONG LIU: The chapter in the lecture notes, yeah. Yeah. OK. Good. Other questions? Yes?

STUDENT: I'm just curious if gluon's included in the free massless spin 1 particles?

HONG LIU: Sorry?

STUDENT: Gluons included in the free massless spin 1 particles

HONG LIU: Yeah. Gluon is also massless spin 1, but gluon, actually, they interact with themselves. And so gluon is a different. So gluon-- to describe gluons, you have to wait for quantum field theory 2. And so the thing about the photon is that the photon don't interact with itself, but the gluons interact with itself. OK.

Yeah, so essentially, we treat everything except gluons. Yeah. Other questions? Other questions? OK, good. So now we can just move to chapter 2. Now we are talking about this theory. OK. So actually, I should not erase it. So now we talk about this theory.

Because this theory describe free particles, so we call it a free scalar field theory. So this is the theory we are interested in. So now we will describe how to quantize this theory. Good. So first, we will quickly go through the quantization of harmonic oscillator, which you should already have done in your Pset. And so we can do it relatively fast.

So the quantization of harmonic oscillator in the Heisenberg picture. So we will see that once we understand this example in the right way, and then quantizing this field theory becomes trivial. And quantizing this field theory becomes trivial.

OK, so let's start with quantum harmonic oscillator. For simplicity, I take the mass to be 1, and to take the frequency to be 1. Yeah. Yeah, let me put the frequency here, omega. OK. Let's take the mass to be 1. And so for this series-- so this is a simple harmonic oscillator, which you have seen it maybe for most of your intellectual life.

And so P will be x dot. It's a momentum-- the conjugate momentum is x dot. And so the Hamiltonian is the P squared divided by 2 plus 1/2 omega squared x squared. And the equation of motion is x dot --double dot-- equal to x squared.

So let's first look at this-- look at harmonic oscillator as a classical theory. So for classical theory, we know how to solve this equation. We just need to solve this equation. So classical solution. Is given by x t equal to A cosine omega t plus B sine omega t. And A and B, just some integration constant.

And for convenience, I can also write it in the complex form. Right here is following equal to a exponential minus i omega t plus a star exponential i omega t. And a is some complex constant. And again, it's an integration constant, I just rewrite the integration constants slightly differently. So these are just integration constants.

So now-- yeah, so this is a complete solution of the problem. So now let's go to quantum. So when we go to quantum, and then we replace this classical dynamical variable, then become the Heisenberg operator, become the quantum operator. In particular, in the Heisenberg picture, and then this operator will depend on time.

And now this equation become operator equation. So now let's-- maybe I should label my equation. So now this star become an operator equation. Now star is the operator equation for x hat. So you have exactly the same equation as the classical equation, but now the interpretation is different. Now the x hat becomes the-- now the x become the operator equation.

So now the solution-- now let me call this star star. So this still solves that equation. So this still solves that equation, except-- so these are just C numbers, because this is a function of t, these are C numbers. But now x becomes-- so now quantum mechanically, so these have become now, become the quantum solution. So now I have hat.

So this still solves the equation. The quantum mechanic equation just becomes a hat. And so this still solves that equation, but these are C numbers. The left-hand side is the operator, and it can only be that A hat and B hat are operators. And also, a must be operators. And the star we replace it by dagger.

So now, say-- now a, you just go to a hat and a star goes to a dagger. A hat dagger. OK, now these are integration constants for the operator equations. So they are just-- now they become constant operators. So they are just constant quantum operators. They're just quantum operators. So they are integration constants for your quantum operator equations.

So now the solution-- so now the-- yeah, so now this is your quantum solution. So this is the form we will often use. You can also use that form, but the equivalent, but this is the form we will often use. You can also, from here, you take the derivative, you can find the P. So again, this will become an operator equation. You take the derivative of x hat-- x, and then you find p, et cetera.

Yeah. So you-- p hat t, you just take the derivative. Then you can just work it out. It's very easy. So now-- so this equation just-- so we already solved the quantum problem because we find the full evolution-- full solution to the quantum operator equation.

Except that we still need to impose the canonical quantization condition. So this is just equal to i. So the standard-- so if you plug in the expression for x and the t-- and the p into here, and then you find that a and a dagger, the commutator is equal to 1. So this is your familiar creation and annihilation operator for harmonic oscillator. For harmonic oscillator.

And now, we can also use the a to build the Hilbert space because a are the-- yeah, because all your operator now-- because x and p-- x and the p are expressed in terms of a and a dagger. So essentially, any operator of this theory can all be expressed in terms of a and a dagger.

And then you can just use a and a dagger because a and a dagger essentially-- they are fundamental building block of your full quantum theory. And then you can also use that to build the Hilbert space. So the Hilbert space is defined by the lowest state is annihilated by a, and then the higher state obtained by acting a dagger on the ground state.

So this is your full theory. So this is your full theory. And so now, now you can compute anything in this theory just with those knowledge, just with those knowledge. So any questions on this regarding the harmonic oscillator? Good, OK.

So let me just summarize. So this is very, very familiar, but let's summarize the rule we have been using. Summarize the steps in context of the harmonic oscillator. And then the same steps can be used to quantize the field theory. Steps of quantization.

So we make it a general. So first-- so the 0 step is that the classical equation of motion becomes a quantum operator equation. Then the first step is to find the most general solution-- to find the most general solution to classical equation of motion-- yeah, just to equation of motion.

And then you go to quantum, and then you just promote the integration constants in your classical solution in 1, in step 1 to constant operators-- constant quantum operators. So this gives the-- then you have the full-time evolution at the quantum level. Now you know how the quantum operator evolves.

And then you impose canonical quantization conditions. So that will tell you the commutators between those integration constant operators just as we do here. And then constant operators in 2, now you know also the commutation relation between them-- among them, and then now can be used to generate the Hilbert space. OK.

So this step 1 to 4 are very general. And if you can do it, and then you can essentially do it-- apply it to any system. Say one-- harmonic oscillator is one degree of freedom, you can apply it two degrees freedom, three degrees freedom. And also to field theory, infinite number degrees of freedom. And now we will apply these to field theory. Yes?

STUDENT: Do you get the finite dimensional Hilbert spaces from this procedure?

HONG LIU: For this procedure, you cannot, but you can get the finite dimension-- yeah, because the finite dimensional Hilbert space don't have the classical analog. So here, we start with a classical system, and then we quantize it. And the quantum system is a finite dimensional Hilbert space. They're essentially intrinsically quantum. And yeah. Like spin. Spin is an intrinsic in quantum, so yeah. Yes?

STUDENT: This question is to do with spin operators, the fact that there are no finite dimensional representations for these dagger operators. Any dagger.

HONG LIU: Yeah, yeah. Yeah, it's just because they don't have-- yeah-- yeah, the reason is just they don't have classical counterparts, yeah. Yes?

STUDENT: It is it always true that the constant operator is sufficient to generate the entire Hilbert space?

HONG LIU: Yeah, because if you think about it this way-- that's a very good question, because-- let's just look at this harmonic oscillator, and then you can try to generalize it. Because they are integration constant over the x and the p. Then any operator in your theory can all be expressed in terms of a and a dagger. And then and then your Hilbert space must be-- you must be able to generate the Hilbert space using them. Because they are the building block of your whole operator. Yeah, yeah.

STUDENT: How do you get the vacuum state? Is there always a generalization of vacuum state?

HONG LIU: Yeah, yeah. The vacuum state here is based on-- is coming from the energy. So once we solve x and p, and then we can write the Hamiltonian in terms of x and p, and then you just look for the lowest energy state. And then you find the lowest energy state that satisfy this equation. Yeah. And then from there, you can find other states. Yeah, yeah.

Yeah, the same thing we are going to-- yeah, the same strategy we are going to use for the quantum field theory. OK. Good? OK, good. So now it become a mechanical. We can just apply this to this theory. We can just apply this to this theory. And now let me add here.

So here, the canonical momentum density conjugate to phi is-- I called it pi before, is just the time derivative of phi. And the Hamiltonian density, you can find it explicitly, is pi squared plus 1/2. And then this is the classical equation of motion. OK.

So we can-- now let's just solve this classical equation of motion. So this equation can be-- so this equation is easy to solve because of the translation symmetry. You can just do a Fourier transform. So we can Fourier transform. So now let's do can-- you can Fourier transform. So 2 prime can be solved using-- OK.

So we can just write phi x equal to exponential minus iEt plus i k dot x. Yeah. OK. And then you can see then this just solves the-- so this is provide a basis of solutions to 2 prime. It's just a plane wave, just a plane wave.

So now when you plug this into there, then you just get the dispersion relation. Each square should be m squared plus k squared. So we'll denote this as omega k squared. So omega k is defined to be just the k squared plus m squared.

So E-- when you take the square root of E-- so you can take plus-minus omega k. Can we plus-minus omega k? So we normally call the solution-- we normally separate-- so for historical reasons, we normally put-- define u k x to be exponential minus i omega k t-- x. OK, now we have inserted the positive root of E. So this is normally called the positive energy solution.

Even though this-- even though this name is a little bit misleading. So actually, this-- we don't define the energy actually-- yeah, later we will see, this is not really the energy of a particle. And so this is just a traditional name. This is just a traditional name. Conventional name.

And then you can define the complex conjugate of k. And now you have-- then corresponding, you have a minus omega k in there. So we take this complex conjugate. And so this is called the negative energy solution. So altogether, they form a complete set of solutions. So complete set of solutions. So complete basis-- it's a complete set of-- complete basis is formed by u k and the u k star for all k. For all k.

So these are-- when you-- these are the complete set of solutions to that wave equation. So that's just the-- yeah. Any questions on this? So this is just like a-- classically, this is like a wave. Just like a plane wave. Which you should also have seen in 8.03. Yeah. Good.

So now we can find that-- so now we can write down the most general. So this is a basis. So these are the constant part of the exponential plus-minus i omega t here. So now we can write down the most general solutions by just putting in the integration constants. So the most general classical solutions.

So you can just write phi x equal to integrate over all possible value of k because this is for all k, so we just sum over all of them. And so we so-- so this factor is just for convenience. OK. It's just a convention. You don't have to put it here, it's just a convention.

And then we have a k u k plus a k star u k star. So this is just the most general set of solutions with a k and a k star as integration constants. So this is a full set of the integration constant. Good?

So now when you go to quantum level, so now we can just follow the rule. We find the most general classical solution. And in the quantum level, we just promote this to be operator. You just put the hat there and change this to dagger. So now this become the basis of quantum operators.

So these are the full set of constant quantum operators. And this solves your theory on-- so this solves the operator equation. And this solves the operator equation. So now the next thing is to impose the canonical commutation relation.

So first, we have to-- now we have to do a little bit thinking. So so far, it's just straightforward, but now we have to do a little bit thinking. So for finite, for harmonic oscillator, or for quantum systems of a single variable, you just have x-- you just have these. You just have these. And now we need to come up with a generalization of these to a field theory.

So we need to come up with a generalization of that field theory corresponding to phi x. So now let me making the time and the spatial coordinates separate. And its conjugate momentum is phi-- is pi. Conjugate momentum density. So we should do them at the same time.

Remember, p is the same evolution operator, so they have to be evaluated at the same time. So it's equal time. Canonical quantization conditions always are the equal time. But x is a label of operators. So x, they don't have to be the same. So hee, it can be x; here, can be x prime. OK. So now we have to come up with a generalization of what is this quantity for field theory.

So now we just need to do a little bit of guesswork. You can easily guess it. So before we do that, do you have any questions on this? Yes?

STUDENT: x is already the operator in this feature?

HONG LIU: Yeah, yeah. x is always-- so x is always-- here, it's always just the label of the spatial location. Yeah, it's a label for the-- yeah, yeah, it's your field theory label. Yes?

STUDENT: So in this procedure, I guess you always end up with your operators as being constant in time. Is there any way that you can get them where it's like the evolution is more complex rather than just a constant operator and phase factor?

HONG LIU: Yeah. So normally, if you have second-order differential equation, you always have some integration constant. Yeah, that's it. Yeah.

STUDENT: So that carries forward all throughout?

HONG LIU: Yeah, yeah. Yeah. Other questions? Yes?

STUDENT: Should it be obvious why the classical equation of motion translates directly into the operator equation of motion?

HONG LIU: Right. Yeah, yeah. So that's a very good-- that's a very good question. So that's just an extension of our usual procedure for the quantum mechanics. So the usual procedure, when you-- even just for the harmonic oscillator for single-variable system, you have this correspondence between the classical system and the quantum system. When you quantize the classical system, then the classical equation of motion become a quantum operator equation.

Here, we just use the same rule because quantum field theory is just the theory of infinite number of degrees freedoms. We are not changing the rule of quantum mechanics. And so that's why we just-- again, just promote the classical equation into the operator equation. Other questions? Yes?

STUDENT: One way of understanding the Heisenberg equations for the quantum mechanics is really the Poisson brackets for the classical theory. Is there something like that for field theory as well? I guess the--

HONG LIU: Yeah, yeah, yeah, there is, yeah. So classically, you can define the Poisson bracket between the classical field variables. And then quantum mechanically, it just become quantum commutators.

STUDENT: Is that how we could come up with those commutation relations?

HONG LIU: Yeah, yeah, you can also do that. That's right. Yeah. So one way to come to this is you first describe-- first, you need to generalize your standard Poisson bracket for finite number equals freedom to classical field theory, and then you can just generalize that to the quantum. Yeah, indeed. That's one route of doing it. OK, other questions? Good?

So we can just guess the answer. The answer is very easy to guess. So remember-- so if you have a single x and a p, that's what you have.

But if you have more than one particles, if you have more than one particles, say just hint. Say you have multiple particles system in quantum mechanics, and then you have x a and p a as your dynamical variable. So a equal to 1 to, say, n, say, the number of particles.

And then your canonical quantization condition just become x a t p b t equal to i delta a b. And that different x a commute and different p commute given the p commute. Some reason, let me just put this out.

So now this a and b are essentially just replaced by x and x prime. So x and x prime are just continuum version of those a and b. Remember that we can't emphasize the x and the x prime are the labels of your degrees of freedom. So now you can just guess-- so we must have the following scene.

So from here, we must have phi t, x phi t, x prime must be 0, and then pi t, x-- so pi is the analog of p here. yeah, so those are operators. t, x prime must be 0. And then phi t, x with pi t, x prime should be something-- can only be 0, when x not equal to x prime, can only be 0 when x equal to x prime. OK, as a generalization of this. And so you can-- now you can guess, so what should this be? So-- yeah?

STUDENT: So direct delta?

HONG LIU: Yeah, this should be just Dirac delta. But now you can ask the question, why has to be direct delta? Maybe why should not be, say, the derivatives of Dirac delta? Say, why should not be, say, 100th derivative of Dirac delta? And that question can be addressed just by from dimensional analysis.

So here, we know-- somehow this must be related to Dirac delta, and now let's decide. So now you can do the dimension-- do a little bit dimensional analysis. So if you just write down the action-- yeah, the action I have just erased. Sorry. So if you look back on the action-- let me just outline the idea because I'm sure you can do dimensional analysis yourself.

So if you look at the action-- so the action is dimensionless in the natural unit we are using. So from that, you can deduce the dimension of phi to be 1 over L-- so 1 over the length. And from the fact that the pi-- where's pi? And maybe I have also erased-- is equal to phi dot, means pi should be dimension 1 over L squared because you take the derivative on time, and then there's another factor of L.

Then that means, on the right-hand side here, must be something 1 over L to a cube because there's no other parameters here. Yeah, because here, there should be an i. And if it's a dimension 1 over cube, there can only be the delta function, not the 100th derivative of delta functions. So this thing should be just a delta function. So this is the convention that there should be i. And then it should be just the free delta function. And this indeed have the dimension of 1 over L cube.

Good? So now, you can just plug-- so you have the expression for x, for phi. You take the time derivative of this, you get the expression for pi, and now you can just plug them into here, you can just plug them into here. And then you can find the commutation relation between those a k's. And so this is a slightly tedious calculation, which is, however, a little bit fun, which, of course, I will leave you to do.

So if you just plug them in, and then you can deduce that the following commutation relation between a's. So this is-- I think this is in Pset 2, but I can still change my mind. Yeah, I wanted to put in Pset 2. So you find the commutator between a and the commutator between a dagger-- yeah, so now, we will suppress the hat because if I write hat I think over and over will be too tired.

So these are 0. So the commutation relation between a and 0, and a-- between-- within a and a dagger. So this gives you 2 pi cube delta function in k. So this is a three delta function k.

So again, this is a straightforward generalization, so if you have multiple harmonic oscillators-- so if you have considered the multiple harmonic oscillators before, and then the a between the different harmonic oscillator because k prime are just-- here, just corresponding to-- essentially you have-- yeah.

Here, this is-- essentially, you have infinite number of harmonic oscillators. And each one of them labeled by a k. So this is just like-- essentially we find-- yeah, let me just write it here.

So from those commutation relation, we conclude this theory, when we quantize-- after we quantize it, become ar infinite number independent harmonic oscillators, decoupled harmonic oscillators. Harmonic oscillators labeled by continuous parameter k. So k is-- yeah. k is the wave number.

So for each k, there is a, and a-- so between a themselves, it's 0; between a daggers, it's 0; but a, a dagger, they not equal to 0. And so this is, again, the continuum generalization of 1. This is a continuum generalization of 1 because you have a continuous variables. Yes?

STUDENT: So how about the commutation relations of different times?

HONG LIU: Yeah, then you cannot say for sure.

STUDENT: Like by Lorentz invariance x and t be on the same plane, shouldn't we have the same--

HONG LIU: Yeah, yeah, yeah. No, but you see the quantization condition is in quantum mechanics. Quantum mechanics, t and x are not on the same footing. You can require your action to be-- x and t to be on the same footing. Once you start quantize your theory, and then t will have a pronounced role.

STUDENT: OK, so if I wanted to, I couldn't write like the commutation relation as functions of for vector x?

HONG LIU: No.

STUDENT: --delta x--

HONG LIU: No, no, no, no, no. The canonical commutation relation have to be imposed at equal time. Other questions? Good? So yeah. So essentially, we just get-- and now it's just trivial. So you can just build up your Hilbert space. Essentially, you just have infinite number of harmonic oscillators. You just have infinite number of harmonic oscillators.

And there's no surprise, you get infinite number of harmonic oscillators because we mentioned that this field theory can be actually written as a continuum limit of these particles on the chain, which, in these 8.03 three examples, you know, that's just a harmonic oscillator. Once you find the lower modes, they're all just a bunch of harmonic oscillators. And this is just a three-dimensional version of that.

And now we will-- yeah, today we are running out of time. So next time, we will see that each excitations of the harmonic oscillator can be interpreted as a spacetime particle. So that's the cool thing of it.

And now you have this infinite number of harmonic oscillator, and now you can act-- now you can define the vacuum, and then act this creation operators on the vacuum. And now you find each excitation actually corresponding to a particle and has the-- corresponding to a relativistic particle.

And that's how you can have actually arbitrary number of particles in this theory. And yeah, because you can excite as many times as you want. Each excitation is a particle. Good, good. OK, so I think it's a good time-- yeah, we are two minutes, I think, early, but I think it's a very good place to break.