[SQUEAKING] [RUSTLING] [CLICKING]

PROFESSOR: Last time, we started talking about the quantize-- the Maxwell theory in the Lorenz gauge. So in the Lorenz gauge, we consider the following action.

OK. So we showed earlier that the Lorentz gauge can be ensured just following the equation of motion. So you just get from here, the equation of motion, and then the equation of motion-- yeah, so the equation of motion will lead to partial square, partial mu A mu equal to 0.

So this is ensures partial mu a mu equal to 0. You just have to make sure your boundary conditions, such as the partial mu A mu equal to O. OK?

And in particular, the action is particularly simple for xi equal to 1, in which case we just have-- actually we just have [INAUDIBLE]. OK.

So as if you just have 4 decoupled massless scalar. OK? As if you just have 4 massless scalar. And the equation of motion is actually all very simple.

So we can just proceed. Copy our result before for massless scalar. And just treat each A mu as a massless scalar, and then we can just do it.

For example, the canonical momentum conjugate to A mu would be just a mu dot, just A mu dot. OK. So the canonical quantization condition, then, is given by-- and we can also just straightforwardly write down A mu, the expansion for-- the operator expansion for A mu. OK.

So we have four of them. So again, we will-- instead of writing them as 4 massless scalar, we will-- as we did before, for the Coulomb gauge case, we introduce a polarization vector. So we will introduce a polarization vector. So there are four possible things. There are four components. So there are four possible polarizations-- times.

OK. So we just get that. So this epsilon mu alpha for alpha equal to 0, 1, 2, 3-- so these are four polarization vectors. OK?

You can choose. You can just choose, say, 1, 0, 0, 0, 0, 1, 0, 0, et cetera. And then this just like four decoupled scalar, OK? But this way introduces a polarization vector, allowing to write them in a more general way.

So we normally pick-- for example, the 0-th component, the 0-th polarization vector just to be along the time direction, OK? So just along the time direction. And then we also introduce another-- so the number 3, we introduce to be proportional to the direction of the momentum.

So this is proportional to the-- parallel to the momentum, to the spatial momentum. So the k mu here would be-- because this is massless. So we just have this. OK so so the 0-th component is just equal to the magnitude of k. And then we take the epsilon 1, 2 epsilon mu, then 1, 2 to be, let's say, orthogonal to the momentum.

So these are called the transverse polarizations. So we take them also to be orthogonal to 0-th component. That means there's no-- the time component for 1, 2 will be 0, too, and then also it should be proportional-- it should also be orthogonal to the momentum.

So we take, also, them to be orthogonal to each other. Eta mu nu. We take them to be orthogonal to each other. This is the normal, orthonormal condition. OK? Take these spaces to be orthonormal.

And then this is also complete. Means that if I sum over different alpha and beta-- and actually I can get back to eta mu nu. OK? So this is called the completeness. So they form a complete basis.

So any Lorentz vector can be expanded in terms of it. So if we take, say, k to be the z-direction, then the simplest choice would be, say, you just do-- so we just have that.

OK, so this is simplest. But you can consider more general polarization. And so now, if you plug in this expansion, as we do, for the scalar, and then you plug in back to that commutation relation, then you will get the commutation relation with a. So this is straightforward. So if we call this equation star-- star star, this equation star.

So plug star star into star, and then we get the ak alpha ak prime beta dagger should be equal to beta alpha beta 2 pi cubed delta -- prime OK. And the the rest commutators are all 0.

And then you can define the vacuum. Define a annihlated by all the alphas for any alpha and k. And I will introduce the Hilbert space. I call it H big to be the psi-- the collection of psi built by acting ak alpha dagger on 0. OK? So you can just add arbitrary number of them on 0, then you get your Hilbert space.

So we call this Hilbert space to be my big Hilbert space. And you will see the reason we do that. OK? So later, we will see why we call this H big. So this is seemingly all fine.

But of course, there are problems. This cannot be right, because the Maxwell theory, we said, only has two transverse degrees of freedom. But here we have four. We have four massless with degrees freedom. We must have too many.

And indeed, you see there, here, are problems. So one problem is the following. So one problem is the following. So there are problems.

So one problem, yeah-- so the zeroth order problem is that we have four massless degrees of freedom, two too many. OK? So we know that we only should only have two.

And the second, you see that this is problematic. Is that, you notice-- so here, we have eta alpha beta eta mu mu. So this is all just from Lorentz covariance, OK?

But this eta alpha beta is problematic because if you look at the commutator for ak 0-- OK, so the zeros-- that polarization, then you have ak 0 ak prime 0 dagger equal to minus 2 pi cubed. Say delta k minus k prime. OK, you have a minus sign.

So this is problematic because if we, say, create particles-- say if we create the particles by acting ak alpha dagger on the 0, and then you find, because of this minus sign, this create the particle with polarization epsilon alpha mu.

So now, because of this minus sign, you see that if we look at the overlap between the particle with the 0 polarization-- so if we look at k 0 with some k prime 0-- and then because of this minus sign, we have minus.

So this will be proportional to minus 2 pi cube theta cube k minus k. So this is smaller than 0.

So you see, actually, in the product-- so that means that this state actually has negative norm. That means this state is negative norm. But this is actually a good sign. So this is another way to see that we have too many degrees of freedom.

So that means not all degrees of freedom here can be physical. So here, it just tells you there are some degree of freedom here that must be unphysical because they have negative norm. The states they create have negative norm. So they cannot really correspond to genuine physical states.

But those problems should not worry us because, so far, what we have done is to quantize this theory. So far what we have done is to quantize this theory. But this theory is not the Maxwell theory.

To get the Maxwell theory, we have to do two more steps. One more step-- one step is to impose conditions so that makes sure this partial mu a mu is equal to 0, because we haven't fixed the gauge. The second condition, we mentioned before. So there are two more things we need to. So first is that we need to ensure the gauge partial mu A mu is equal to 0.

The second thing, as we mentioned before, is that after fixing the Lorenz gauge, they're still residual gauge degrees of freedom left. And we have to fix the residual gauge freedom in the Lorenz gauge. OK?

So when you do this, then you will get a physical Hilbert space. You will get the physical Hilbert space. And this physical Hilbert space, we will see-- you will see-- will only contain, indeed, two transverse massless degrees of freedom. And the two degrees of freedom here are gotten rid of, I got rid of.

And so in the past, I described those procedures in detail in class. But actually, that didn't have very good effect. So later, I put it in your Pset. Actually, I think it worked better, so forced you to go through it and to think through it, how to do these two steps. Because those steps, they are not difficult technically, but you need to think carefully. You need to think carefully.

So let me just make some comments on this step. So recall in the Coulomb gauge-- so in the Coulomb gauge-- so recall in Coulomb gauge, we impose this condition. We essentially solve this condition. We impose this condition.

So classically, we impose this condition, essentially, as part of the equation of motion. And we solve the A, which satisfies this condition. We just one into two transverse A, OK?

And then quantum mechanically, because we impose it as a classical equation-- we impose it as part of the equation of motion, classically. So quantum mechanically, this becomes an operator equation, which we impose on operators. But in this case, we can no longer impose.

But in Lorenz gauge-- so this leads to the operator equation. But in the Lorenz gauge, we cannot impose partial mu a mu as the operator equation, because the equation of motion already almost implies this is equal to 0.

We just have to impose some boundary condition to ensure that, indeed, this equation only has solutions corresponding to partial mu and mu equal to 0. So if you impose this separately as the-- yeah. So you only need to impose them as boundary condition. So quantum mechanically, that implies we cannot impose this as an operator equation. Indeed, you are showing your pset, and it will be inconsistent if you impose this as an operator equation.

So what does it mean at the common level corresponding to classically you impose the boundary condition? So classically-imposed boundary condition limits possible configurations you can have, limit possible configurations you can have at the quantum-mechanical level corresponding to restriction on the states. So the quantum level, it turns out that the right thing to do is to impose this kind of condition on the states, require your states to be annihilated by something like this.

But it turns out, actually, the story is more subtle than that. If you just require the state annihilated by this thing, actually it does not work. And so you have to do a little bit more subtle. So the fun will be in your Pset. You will go through that. You'll go through that. So do you have any questions? Yes?

AUDIENCE: Why is it too much to just say that is 0? I guess it makes sense that you can just fix the boundary conditions but --

PROFESSOR: Yeah, so classically, it does not matter. As a classical, it doesn't matter. But quantum mechanically, you will show you a P set if you impose this as an operator equation. Then that's incompatible with this canonical quantization condition.

AUDIENCE: Yeah.

PROFESSOR: Yeah, here, I just motivate that classically in order to ensure this unity to impose the boundary condition, not as the equation of motion. So quantum mechanically, we just-- yeah. Other questions? Yes?

AUDIENCE: Is there a reason why the subspace with positive norm is like closed -- a closed Hilbert space?

PROFESSOR: Yeah, you will see. You will see. You will see what's happening, yeah. So it turns out-- yeah, let me just say some words. So it turns out when you do this step, and then you will eliminate-- so the step one, we eliminate the negative norm state.

AUDIENCE: OK.

PROFESSOR: So you find that once you impose properly this kind of condition on the states, and you actually eliminate the negative norm state. But then you find after you eliminate the negative norm states, in this Hilbert space-- in this Hilbert space, there's still states of zero-norm. There's still states of zero-norm. And then when you fix the second, and then you eliminate those zero-norm states.

And then you get your physical Hilbert space, because, remember, in the Coulomb gauge, everything is just like harmonic oscillator. There's no zero-norm state. You fix the gauge completely.

But here, because you didn't fix the gauge completely-- and even after you eliminate those negative norm states, you have zero-norm states. And then they correspond to remaining gauge freedom. And once you get rid of them, and then you have your-- yeah. Good. Any questions on this? Yes?

AUDIENCE: Instead of doing Maxwell theory, what if we're just trying to do four massless scalars?

PROFESSOR: Yeah.

AUDIENCE: And if you try to do it with these vectors, you'd still get these negative norms. What's going on there?

PROFESSOR: Good, good, good. That's a very good question, which I'm waiting for you to ask. So the key thing-- so you say we can certainly consider four massless scalar field. If I consider four Maxwell scalar field, there should not be a problem, why there should be negative norm state or zero-norm state. But the key is that here, the states are contract-- this A mu A mu are contracted using the Lorentzian metric.

And so if you look at this Lagrangian, the 0-th components actually have the opposite sign to the standard massless scalar field, and because of that. So this is not your ordinary massless scalar field. They actually have opposite signs in your Lagrangian. But you say, oh, can we just change the sign for that component? You cannot because we have Lorentz symmetry.

Because we have Lorentz symmetry.

PROFESSOR: So the Lorentz symmetry forces you, somehow, when you reduce the Maxwell theory to this four massless scalar, one of them have the wrong sign. So that's why-- so that the wrong sign is related to why you have the wrong sign here. The same kind of Lorentz covariance tells you here is eta alpha beta rather than delta alpha beta, and then delta alpha beta. Good? OK.

So this concludes our discussion of the canonical quantization for the Maxwell series. So here I will quickly describe how to do it for the path integral. So I will just do it very quickly, because many of the elements are familiar before. And so I will just point out things which are new for the Maxwell case. OK?

So we already familiar how to do the path integral. So Maxwell theory is a free theory. So in principle, we know how to do it. So let's just-- yeah. So here, let's go back without fixing the gauge first, OK? So this is now completely starting from the original, just starting from the Maxwell equal to minus 1/4 as with mu F mu mu. So let's just go back to this theory.

Let's go back to this theory. And then let's just try to do a-- try to see how the path integral works. So remember, this is a free theory. So we can just write down the generating functional for this theory. And then we can, in principle, calculate all possible correlation functions here. Free theory is supposed to be very simple.

So we just write down the generating functional, which we integrate over all A mu. And then we have i S-- so this is Maxwell theory-- then we add the source. So this J mu will have nothing to do with electromagnetic current. This is just a J mu with the generating function used to take derivatives. So this is just an external source.

So now this is a Gaussian integral. So in principle, we can just do it. Because this is quadratic, and now we are familiar how to do the quadratic integral. We can just do it.

And to do it inside, we always write it in the form like this. We write it in the matrix form. OK, we write it as a matrix form. And this K mu nu just can be read from here by integration by parts, et cetera. So let me just write down the answer.

So this gave me mu just equal to-- so let me call this 0, call this 0 K mu nu. So this just given by partial square eta mu nu minus partial mu partial nu, and then the delta function. And the delta function. So this partial will only act on x.

So naively, we can just-- so now this is just a Gaussian integral, so we can use principle just to directly write down the answer. Actually, let me keep this here.

So in principle, we can just directly write down the answer, just this Z J equal to some number, some infinite number, which we never care, and the exponential i d 4x d 4y then J mu x. Then the K 0 mu nu minus 1.

Yeah, so let me just write it like this. K0 minus 1 mu mu. x minus y J mu. OK?

So this is just almost exactly the same as we do before. So this is like the kernel for your Gaussian integral. And you just take the inverse of it. And then you contract it with the source.

OK, so you just know the only difference from the scalar case, or the spinor case, other than the functional space of x and y, now you also have a matrix in terms of Lorentz index, mu nu. Everything is the same as before.

So it looks like, then, we already solved this theory, OK? But you should feel uneasy. You should feel easy because when we quantize it using the canonical method, we do have to go through some trouble.

So how come, somehow, path integral just do it immediately? Somehow the trouble has to be conserved, no matter what method you use, no matter what method you use.

Indeed, can someone guess what would be the potential trouble here? Yes?

AUDIENCE: Are you overcounting, because when you integrate over, like, DA, you could have two-- two A's differ only by gauge?

PROFESSOR: Yeah, right.

AUDIENCE: Yeah, so then you're integrating over too many?

PROFESSOR: Yeah, indeed. That's the correct statement. But if I just do it straightforwardly, somehow, then what's wrong with that? So, yeah, what you said is conceptually-- conceptually, you should suspect there's something wrong here. So whenever you conceptually suspect there's something wrong here, then you should technically be looking for some mathematical problem, because the conceptual mistake will always reflect in some mathematical difficulty. Yeah. Yes?

AUDIENCE: So you have to integrate to measure [INAUDIBLE]

PROFESSOR: Yeah. Yeah, you're right. It's just you integrate more things. You integrate more things, then you just get more infinite constant, but we don't care.

AUDIENCE: Maybe like a sign will be off, because the sign was off earlier, right? So when you do the eigenvalues of the determinant or whatever, maybe it messes that up.

PROFESSOR: Yeah. Yeah, but we have i, so it doesn't matter the sign. Yes?

AUDIENCE: Does the K end up not being invertible or something?

PROFESSOR: Yeah, exactly. So the key is that the K is actually not invertible. Turns out K minus 1-- so K 0 actually is not invertible.

So the reason is simple. It's that because the gauge freedom in the original theory-- because the gauge freedom in the original theory, by gauge freedom, we mean that when you make a transformation by some arbitrary function, your action does not change.

Then that means that this kernel must be invariant under some transformation. It's invariant under some transformation means that this must have zero eigenvector. When it has zero eigenvector, then they must not be invertible. OK?

So now let's just see it explicitly.] To see it explicitly, it's easier to see this in momentum space. So if we see this in momentum space-- so in momentum space, we can just directly write down the K0 mu mu. It's just equal to k square, just k mu k nu minus k square eta mu mu.

So now it's obvious this has a zero eigenvector because this is precisely the minus k square, the transverse projector. So P mu nu T is the projector into the direction perpendicular to k mu. So this satisfies the property that pT mu mu is equal to eta mu mu minus k mu k mu divided by k square, and satisfies the property p mu mu T k nu is always equal to 0. OK?

So you can easily see it, because if you contract with this k mu, and k mu contract with here and give you k square, and cancel with k square here, and you have k mu, and then k mu contract with this one, you just get k mu. So they get canceled. So this is the projector to the transverse space. Of course, it has zero eigenvalue-- zero eigenvectors.

So then we see that this is not invertible in momentum space, and then it will not be invertible in coordinate space. It will not be invertible in coordinate space. And now, in coordinate space, now we can see with this understanding, then now we can immediately see in coordinate space the eigenvector of zero eigenvalue precisely corresponding to a gauge transformation.

It's because in coordinate space, k mu translate-- when you translate to coordinate space, means partial mu. So k mu to coordinate space means partial mu. So that means that for any function, partial mu lambda-- this is K0 mu nu, partial nu lambda must be 0.

And you can check explicitly. So if you contract with partial mu here, and then you have a partial square and you have partial mu, and then eta mu mu will give you partial mu, again, they cancel.

And of course, this is just precisely the gauge transformation. This is precisely the gauge transformation. OK, so the gauge transformation precisely is the zero eigenvector of this kernel. It's the eigenvector of zero eigenvalue of this kernel. So that's why this is not invertible.

So the way to fix is simple because remember this picture. So this is the space of the full configurations of A mu. And A mu along those trajectories are equivalent. To quantize it, we need to just-- the physical configuration corresponding to a section of it. So what this says is that K0 is not invertible in this full space because you have zero eigenvectors along this direction, OK, zero eigenvector corresponding to this direction.

But if we restrict to a cross-section, and then this zero eigenvector no longer exists. And then it will be invertible. OK, so once we fix the gauge, then we expect-- then the k mu after we fix the gauge should be invertible. Good?

So now let's fix the gauge. So K0 mu nu restricted to a section should it be invertible. OK, so we need to fix the gauge. So we will do this for the Lorenz gauge. You can do similar things for the Coulomb gauge.

The reason we do the Lorenz gauge, just the Lorenz gauge, later when we introduce interactions, we will always work with the Lorenz gauge. And when we discuss QED, we will always work with the Lorenz gauge. So here I will elaborate in the Lorenz gauge. Yes?

AUDIENCE: Can you explain what you mean by when you do the cross-section, the eigenvectors here doesn't exist anymore?

PROFESSOR: Oh, yeah. So this zero eigenvector-- so this is a zero eigenvector of the K0, right? So at the partial mu, partial mu lambda, just parameterize this direction. But if you restrict to a section of it, and then you are not allowed to worry away from it. And then this direction just gone, so you no longer have that zero eigenvector anymore. Yeah. Yes?

AUDIENCE: Should there be a convolution in the equation on the bottom left?

PROFESSOR: Yeah. Yeah, good. Indeed, you have to integrate-- yeah, you have to integrate over y. Yeah. Thank you. Yeah, if I write this precisely, you have to integrate over this y and y. Yeah, that's right. So this is the proper way to write it. Good.

AUDIENCE: Why does this issue only happen for gauge symmetries? Like, why don't the other symmetries need to do the same?

PROFESSOR: Yeah. Because other symmetries-- you have local symmetries. So the local symmetry tells you there's really some degree of freedom redundant. So this is really corresponding to-- so this lambda x, so all possible choice of lambda, is really corresponding to some finite trajectory in your configuration space.

So if you have a global symmetry, global symmetry is independent of coordinates. So from this point of view, a global symmetry just relates to the configuration at this point to the configuration at that point and at that point.

A global symmetry, because it's independent of spacetime, that transformation doesn't change your number of degrees of freedom. They don't correspond to actually a trajectory in your configuration space. Good. Other questions? OK, very good.

So now we will fix the gauge. So in the path integral, to fix the gauge, in principle, is straightforward. So we will consider Lorenz gauge.

OK. You can do the similar thing with the Coulomb gauge. Just Lorenz gauge, normally, we work with more. So how do we fix the gauge? So in the path integral, we do it easily. So we have this integral. Then we insert the delta function corresponding to the Lorenz gauge.

So this should be considered as a delta function. So this is a delta function at one point, and then you take the product over all points. So this is the delta function in the functional space.

And then we will have i S, this S Maxwell, and then you can have A J. You can consider the generating function. So this will restrict you on some cross-section, on some cross-section. OK.

So let me make a remark here. So here, you actually have to be a little bit more careful. So what we actually really want to do-- OK, what we really want to do is we just want to insert here. We just want to insert here delta A mu equal to A mu gauge fixed.

What we want, what we really want, to do is we want just to restrict A mu to some cross-section. But for example, here, for the Lorenz gauge, we can normally not solve this condition. So that's why we do here, because we don't know how to write this a mu fixed explicitly. OK? But going from here to here, actually there's a non-trivial Jacobian, because this is a non-trivial in the functional space. And then this is a non-trivial function on A mu.

So in principle-- the related-- in principle, there is a Jacobian. So let me just write this explicit. A mu x are related by some Jacobian, then delta A mu minus delta A mu fixed-- uh minus A mu fixed, OK?

But now, notice that this function is a linear function of A. So remember, when you do the Jacobian, you have to take the derivative. Yeah, just remember this function. Remember this formula. Delta f x is equal to 1 over f prime x delta x minus x 0. So x0 is a solution for f x equal to 0. OK?

And so you have-- yeah, so this is a Jacobian when you convert the delta function. If you have multiple variables, then this would be the Jacobian. So this course, you take the derivative on this function.

But this is a linear-- this thing is linear in A mu. When you take the derivative, you will get something which is independent of a mu. So this Jacobian will just involve in some-- it will be independent of a mu in the function space.

Something independent of a mu is just some constant. So this just gives you some infinite constant. And so we always throw infinite numbers infinite number of ways, so we don't care about it. And so we can just write down this. So this is just a remark.

So now we have to evaluate this guy. So now we have to evaluate this guy. It turns out that this guy is still not easy to solve. And for the same, it's not easy to do, just for the same reason. We don't know how to solve this A mu fixed. OK, we don't know how to solve this condition explicitly.

So now I'm going to use some trick. I will use two tricks. OK, I will use two tricks to convert this into something manageable, to convert into this manageable. And so this trick applying to the Maxwell is like killing a little bird with a big cannon. And so, yeah, it works here, but it's real, genuine use is actually in the long Abelian gauge theory. So when you quantize the Yang-Mills theory, that becomes essential, because it will be very complicated quantizing using other methods.

And so quantizing Yang-Mills theory become essential. But nevertheless, let me just tell you these two tricks, how to treat this here. OK?

So this method is called Faddeev-Popov method, and it was invented by two Russians, former Soviet, Faddeev, Popov, in the '60s. So actually, there was a story behind it. So when they invented this method, they actually wanted to quantize not this Maxwell theory, this non-Abelian Yang-Mills theory, which later will become the standard model. It becomes standard model of particle physics, electroweak theory, QCD, et cetera.

But when they worked on Yang-Mills theory, Yang-Mills theory was some small corner of mathematical physics. Nobody cared. So when they invented this method, nobody really paid attention. Nobody really paid attention. But then, in 1971, 't Hooft, who was a 21-year-old, like a graduate student, then he used that, which quantized the non-Abelian gauge theory coupled to Higgs.

So before that, people thought that theory was inconsistent. But then he used this path integral to quantize that theory. And so that was a triumph, for which 't Hooft got a Nobel Prize a number of years ago. And when 't Hooft came out using this method, nobody could understand it in US. People like Weinberg, who wrote the electroweak theory, which 't Hooft quantized, he couldn't understand it. And there's only one person, one person who could understand it in the US.

He was an assistant professor at Stony Brook. So he started his assistant professor going to Stony Brook, and in Stony Brook, there was a big shot, the Nobel Prize winner, C.N. Yang, who invented this Yang-Mills theory. So he was asking Yang, oh, what should I do for my research, ask for a Nobel Prize winner for advice. And Yang said, maybe you can look at this Faddeev-Popov stuff. And then it turned out later only himself could understand 't Hooft's paper. And then he made it-- of course, he also did some other good stuff.

But immediately, he became the only person everybody went to because he really understood the story. Yeah, anyway. Yeah, so it's an actually very interesting story behind this.

So first, we have to actually modify this Lorenz gauge condition a little bit. So rather than writing as partial mu A mu, let's try to impose the gauge partial mu A mu equal to some arbitrary function Bx, because partial mu a mu will give some trajectory here. But if I choose some arbitrary function Bx, it will just give you some other trajectory. It's still some trajectory.

So what Bx you choose doesn't matter. So you can just replace this delta function partial mu A mu here by delta partial mu A mu Bx. So this is the first non-trivial step. OK? So this will give you some other trajectory.

So now, since the Bx does not matter-- OK, since the Bx does not matter, we can actually integrate over Bx. And then the next step is to-- so first you replace that. So this is the first step.

And then the next step is that you replace Z by you integrate over Bx with a measure B squared x, and then original Z. So original ZJ. We can now depend on B because I have replaced this condition.

And now since this B does not matter, so I can just integrate over B with some arbitrary measure you want. OK? The only thing this does is just give you an infinite constant. Again, we don't care-- infinite constant. OK, so this gives you some infinite constant since this thing is supposed to not be dependent on B.

But this, actually, is very useful because the reason we do this is because now, when we do this, the path integral will become the following, become DB DA mu. Now we have delta partial mu A mu minus Bx.

And then we have exponential i S, then minus this i xi 1/2 B square. And then I have this i J dot A. So now you see the benefit, because now I have integration over B. But the appears very simply in the delta function.

So now I can evaluate the delta function by using the integral of B, by using the integral of B. So now I can just straightforwardly do the B integral.

So now I get a new action. i S xi plus i J dot A. And now this i S xi-- so this S xi just given by your original Maxwell, then minus xi over 2 partial mu a mu squared. OK?

So this is the reason we put B square here, is that when we evaluate the B integral, then we get something quadratic. And now we have that. And now you see this action is precisely this action we have there. OK, it's precisely that action given there. So we also derive it using path integral. We also derive it using path integral, using this trick.

So now we just have this. Now we just have this. And now we can straightforwardly calculate-- now we can straightforwardly calculate this Gaussian integral. One second. Yeah, now we can just straightforwardly calculate this integral.

By the way, the reason I say the story here is trivial, it's a little bit of killing a bird with-- it's because for the Maxwell case, this Jacobian is actually trivial. It's a constant. But if you do it for a non-Abelian gauge theory, say, for the electroweak theory or for QCD, this Jacobian is highly non-trivial. And part of this trick is also how to treat this Jacobian, which we don't have to do here. So here we have a much simpler story.

So now we can just look at this path integral, which now S xi become invertible, because we already know from here this is invertible. At least for xi equal to 1, given by this, this is invertible. So you can actually now invert.

So now we can write this S xi is equal to 1/2 d4x A mu x K mu -- K mu nu xi x minus y A nu y. And this K mu nu-- yeah, K mu nu just whatever you get from that thing, just whatever you get from that thing. And then the K mu mu now is invertible.

And then you find this thing just becomes some constant, then exponential i d4x d4y J mu x, then K xi minus 1 mu nu x minus 1 J nu y. OK?

And again, this inverse gives you the Feynman function. Again, this gives you the Feynman function. So the K xi minus 1 mu mu-- or mu nu. Yeah, so we're not going into detail here. The story is straightforward.

So this should correspond to the Feynman function for the two gauge fields. OK, this should correspond to the Feynman function of these two gauge fields. So now let me just write down the answer. So I urge you to check it yourself explicitly.

So you can easily write down what is K mu nu here. And then you need to write down what is K mu nu here, and then you can write down what is the K minus 1. So let me just write down the answer, which we actually will use later for the Feynman diagram calculation.

OK, so you can show this given by a momentum space minus i k squared minus i epsilon. Remember, now, here it's massless. And you have eta mu nu minus-- plus xi minus 1 k mu k nu divided by k squared. So you can work out the inverse given by this. And so this can also be written in terms to the minus i k squared plus i epsilon.

You can write it in terms of this transverse projector, k mu mu t, then plus xi P mu nu L. So P mu nu L is the longitudinal projector. You just define it by k mu k nu divided by k squared. That's divided by k squared. So this is an answer very simple. So now the interesting thing-- the two interesting things here-- so let me just make some remarks.

So first, if you look at xi equal to 1, then this term vanishes, and then you just have eta mu divided by k squared minus i epsilon. So that's exactly what we expect from here. So here, you just essentially have the massless 1 over k square propagator for the massless particle. And then you have eta mu mu from the Lorenz signature. So for eta equal to 1-- xi equal to 1, indeed we covered that. So comment is that when xi is equal to 0. so this is just proportional to the transverse projector. To the transverse projector.

But remember, the original Maxwell action, we used xi equal to 0. Actually, that's not invertible. So here, we're actually doing a different order. We first take a nonzero xi. We invert it. And then you can set actually xi equal to 0. OK, that's still works. And then xi equal to 0 still works, because we already fixed the gauge. We already fixed the gauge.

So this is the second remark. And then the Wick's theorem for A mu just immediately follows from here. Just as in the case for the scalars and the fermions, it immediately follows. You just contract. We all get these Feynman functions between them. Yes?

AUDIENCE: How difficult would it be to show that the path integral approach and the canonical approach are equivalent for the vector?

PROFESSOR: Oh, no, it's not difficult. One simple thing-- this one point I'm going to make a little bit later, in a few minutes later, and actually for-- yeah, let me make some comment a little bit later. Yeah, they are completely equivalent.

AUDIENCE: It just seems a little not obvious at first glance from the gauge.

PROFESSOR: Right, right. Yeah. So for what we want to do, it's actually a little bit simpler, for reasons I'm going to mention. Other questions? Yes?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah, it's because a projector is just a very simple way to-- so this is just-- so by definition, this is symmetric, in mu nu. This definition is symmetric in mu nu, because under time ordering, you can just exchange them. That means that in terms of Lorentz indices, you must be able to expand it in some symmetric tensors built from k mu. Because k mu is the only vector here.

So that means you have to build this symmetric tensor from just eta mu mu and k mu. And the transverse and the longitudinal projectors are just the tensors you can build from k mu. Yes?

AUDIENCE: Until we fix the gauge, we still have these parameters xi we still have a family of actions. Is that the residual gauge freedom?

PROFESSOR: No. No, that's not the residual gauge freedom. Just so you can show that the physics are independent of xi. Yeah, you can take xi to be any value. OK, good. Other questions? Yes?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Good, good, good, good. Yeah, so that's the comment I'm going to make later. Yeah, that's coming. That's a good question, which I'm going to make later.

Yeah, let me just make this comment and then we'll be clear. I need to make one comment first. So now, for various physical processes which we will consider later, we are interested in the following thing.

So now we can just generalize immediately. Now we can just consider an interacting theory of A mu psi and phi. OK, so A mu will be the Maxwell. This will be some Dirac field. This will be some scalar field or even multiples of them.

So for any such theory, we know how to compute the vacuum correlation function. Then just what we did before just immediately follows, just D phi then x exponential i S phi, and then D phi exponential i S.

OK, so again, I always use omega to denote the vacuum of the interacting theory. And then, again, you can rewrite this as a theory of free theory. Now this becomes, in the free theory, 0 Tx exponential i interacting part 0, and the 0 exponential T SI 0. So now, again, for any interacting theory, we can just use the same procedure, and then we can just do Feynman diagrams. You can just do Feynman diagrams. OK?

And then the Feynman rule-- we can just do Feynman diagrams. And the Feynman rules just follow. So now, as you already asked, this sounds a little bit too simple, because we just look at this action-- we just look at this action, but we haven't done what we promised you would do in your Pset.

Because by doing this, we say you haven't-- you really have not completely fixed the imposed partial mu A mu yet. So remember, we said that when you have this, that will lead to this kind of equation. You still have to impose the boundary conditions to impose the partial mu A mu equal to 0. But now it seems like we are not doing anything like that. We are not doing anything like that. And also, it seems like we are not fixing the residual gauge freedom.

Why don't we do it? So we don't have to do it because, here, we are only interested in calculating the vacuum correlation functions.

And the vacuum is automatically a physical state. As you will see in your Pset, the vacuum is automatic physical state. And so it's already a physical state. So in the vacuum, all those unphysical degrees of freedon just automatically will decouple. You don't have to worry about them.

And so that's why-- but if you're interested in the excited states, then you have to go through the exactly the same kind of procedure to get rid of-- to do a more complicated thing, as you would do in your Pset. But here, since we are computing just a special class of physical quantities, then just by doing this, it's already enough. You don't have to worry about other subtle stuff. You don't have to worry about other subtle stuff.

OK, so I think we are done for the quantizing the Maxwell theory. So now we have all our elements. Yes?

AUDIENCE: So when you have like multiple internal vertices, we used to-- the first interacting theory, when we had just expanded the [INAUDIBLE].

PROFESSOR: You do the same thing. You do the same thing, right? You just expand them.

AUDIENCE: --each coupling, or like --

PROFESSOR: Just expand S i. Whatever is in S i, you just treat them together.

AUDIENCE: I guess, I think, like, earlier in order of whatever [INAUDIBLE] whatever.

PROFESSOR: Yeah.

AUDIENCE: But now is it order of which--

PROFESSOR: Yeah, yeah. Indeed, so normally, we assume all the coupling S I are of the same order, and then you just expand the S I and then-- yeah.

But indeed, you can see the special situation. You may want to expand some couplings, not expand some other couplings. Then that depends on specific situation. Other questions? OK, good.

So let's conclude our discussion of quantizing of Maxwell's theory. And so now-- so we have quantized the photon. We have quantized the Maxwell's theory using the Coulomb gauge. Then we see the two-photon degrees of freedom. And then we also did canonical quantization for the Lorenz gauge, which you will finish in your pset. And then we also discussed how to treat it in path integral so that you can calculate these kind of questions, so that we can treat these kind of questions.

And so now, the next goal-- now we have all the technicality, all the tools we need. Now we can tackle QED. So now, finally, we can tackle QED. So this is the theory of photon and the electron. This is the theory of photon and electron, and so we can consider the physical process in this theory, which A mu and psi are now interacting with each other. A mu and psi are now interacting with each other. So we will start doing it.

I think we are out-- yeah, we only have two minutes left. So maybe we will finish a little bit early today. OK, yeah. So that's all for today.