[SQUEAKING]

[RUSTLING]

[CLICKING]

HONG LIU: Let us start. First, I forgot to mention last time I will always use the metric convention for Minkowski spacetime to be given by this one. So the time component-- I have minus 1. And then the spatial component is positive 1.

And yeah. So also, let me just first remind you what we did in last lecture. So in the last lecture, we talked about the principle of locality, which is a powerful principle. And so this naturally leads to the concept of fields. And so I will generally denote fields-- say, using this kind of notation. Yeah, and when we talk about fields abstractly, we will use this kind of location.

And so phi_a is just a label for different fields. And also-- so the fields depend on spatial coordinates and also depend on time. So the field coordinate-- you can think of this spatial dependence as the label for the fields-- for the degrees of freedom. So at each spatial point, the degrees of freedom-- and the time describe its evolution. OK.

So essentially, this is like-- so classical field is like classical mechanics degrees of freedom but now with the infinite number of degrees of freedom. OK? So the classical field theory is like classical mechanics but now with infinite number of degrees of freedom. Because now you have some finite number degrees of freedom for each point in spacetime. OK? And the principle locality also implies that the action for such dynamical variables-- so the fields are our dynamical variables-- and also implies that the action has a local form.

So the action have the form that is always a single spacetime integral and some function, which is called Lagrangian density, which is a function of the fields-- say, phi at some point at the point x, which-- and then also its derivatives. And so because of locality, then the action has a very simple form. OK.

And then, as in classical mechanics, you can introduce canonical momentum, conjugate to your dynamical variable. So we have momentum density. So you conjugate to phi a each point.

OK? So this is the phi dot, which is the time derivative of phi. And then you can also introduce the Hamiltonian density, which is the pi a phi dot a. So a is summed and then minus L. OK? And then the Hamiltonian is obtained by integrating this over space. Yeah. And also, the equation of motion-- when you do the variation of the action and you find the equation of motion, it's given by the following general form.

OK? So the equation of motion is given by this general form. And when we study field theory, as we study, say, any other subject, we always start with simplistic examples. OK? And the simplest example would be, say, a single scalar field, OK, if you can see the single scalar field, which you have low index. We just have phi, which now depends on spacetime point x.

And then the simplest action for such a scalar field-- you can just write down based on a general principle-- say-- yeah. So you kind of have this form. So we discussed last time the simplest action for scalar field have the following form-- partial mu phi, partial mu phi, and then plus m-squared phi-squared. And then the sign here is determined because of the metric convention. OK.

So this is the simplest scalar field theory we can write down. OK? So here, I have written here a quadratic function for phi, which we wrote as a general function last time. And here, the kinetic term comes from the Lorentz symmetry. So this is the simplest derivative term you can write down which respect Lorentz symmetry.

You can have more complicated terms. You can have this term, for example, squared. OK? But that would be more complicated. So this is the simplest one. And so this will be the simplest field theory we will study first. OK? And yeah.

So in this case, for this example, the momentum density just equal to phi dot-- just time derivative of phi. And then the equation of motion is given by a linear equation phi. It's given by that. So this is a very simple theory with very simple equation of motion. But later, we will see-- I actually will teach us a lot about general field theory.

So it's a very quick summary of some of the main points of last lecture. Do you have any questions? Yes?

AUDIENCE: You haven't specified if phi a is either complex or real.

HONG LIU: Right. So phi here is a real field. Yeah. Yeah. So phi just take a real value. Yeah. Other questions? Good? OK. So now we will talk about the second topic related to our discussion of classical field theory, the symmetries and conservation laws.

So when we have a theory, say, with such an action, we say the theory has a symmetry. So the definition of the symmetry is the following. So symmetry is some transformation of each field phi a into, say, some function of phi.

So this new phi a can, say, depend on all possible fields. OK? And so some transformation of your dynamical variables-- so here, prime just means a different function, it does not mean derivatives, OK-- which leaves the action invariant. OK?

So whenever happens-- and we say this is a symmetry. So for example, in this example, OK-- let me give you a label to this equation. Let me call this equation star. So in the example of star, say, for example, the symmetry includes, say, translation.

OK? So if you imagine we take a coordinate transformation and we do a shift in coordinates, a constant shift-- OK? Let's imagine we do a constant shift of the coordinates. And a mu is a constant. So a mu are constants. OK? So it's a constant vector.

And now if we assume that the phi transform as the following-- so when you change the coordinates, it essentially just changes the label. OK? It changes the label on the fields. But the value of the field should not change. So that means that the phi evaluated at the original point should be the same.

So that means the new transform of the phi evaluated at the new point, OK-- new x prime point-- should be the same as the value at the original point. OK? So this is the-- yeah. So suppose the phi transform like this. So the scalar field should transform like this under a general-- so this is the example of a spacetime transformation, OK, and the general spacetime transformation. as, say, this equation. OK?

So you change the label. But the value of phi should not change. OK? So the new phi evaluated at the new location should be the same as the value of phi at the original location. OK.

And so you can easily check it yourself. OK? So you can easily check it yourself. But I will not do it here. So this exercise for yourself-- that actually this action is invariant under this transformation with this change of x. OK. Any questions on this?

So second transformation-- which this is invariant-- is a Lorentz symmetry. So Lorentz transformation is another kind of spacetime transformation on the spacetime coordinates. If you take x mu to x mu prime, you go to lambda mu nu x nu.

So lambda mu nu is the constant Lorentz transformation matrix. OK? OK? It's the constant Lorentz transformation matrix. And again, under such a coordinate transformation, phi-- will transform this way. Phi should transform this way.

And now you can check it yourself, OK-- and because of this contraction of partial mu, partial mu. And then this is actually a symmetry. OK? Again, this is left as an exercise for yourself. And in fact, I think in your Pset problem, you will do something similar. Any questions on this? Good?

So examples like this kind of transformation for translation and then for Lorentz transformation-- so this is what we call the continuous symmetries. So continuous symmetries-- OK? So these are the transformations-- yeah, I should also give a label to this equation. So let me call it just star-star-star.

So-- [LIGHT CHUCKLE]-- continuous symmetries are transformations, like star-star-star, involving continuous parameters. OK? And so both one and the two are clearly continuous symmetries. Yeah, they are continuous symmetries.

So you can easily see from here-- so here, the transformation is a. And a is four numbers. OK? It's a vector. It's a four vector. And a can be arbitrarily changed. So it's a continuous parameter. OK? You can take it to be 0. You can take it to be 0.1 in all directions, et cetera. So it's something that can be continuously changed.

And similarly, the Lorentz transformation contains continuous parameters. So do you remember how many continuous parameters does the Lorentz transformation contain in four dimensions? Yes?

AUDIENCE: 16.

HONG LIU: Hmm?

AUDIENCE: 16.

HONG LIU: No. [LIGHT CHUCKLE] Yes?

AUDIENCE: Six.

HONG LIU: Yeah, six. It's because you have three rotations. Because you have three spatial directions, you have three rotations. And then you also have three posts-- yeah, so altogether six. So here, there's six continuous parameters. And here, there's a 4. OK? So here is 4.

So both are continuous symmetries. OK? And in contrast, this Lagrangian have-- this action has a lot of symmetry which is not continuous. Can you say what that symmetry is? Good. Priority is a good answer. Yeah. But there's still something slightly simpler. Yeah?

So notice that here, it's all quadratic in phi. And what do you observe? So here, there is symmetry. A lot of the symmetry is phi. It goes to minus phi, OK, because it is quadratic in phi.

And also, of course, there's a symmetry called a parity, which in the case of x goes to minus x. So the spatial direction x goes to minus x. And then phi, again, transforms as this. OK? And then you can also check-- this is also a symmetry.

So in both of these cases, they don't involve continuous parameters. And so these are called discrete symmetries. OK? So there's no continuous parameters. And yeah.

In this case, both of them are called Z2. And this Z2 means that if you do the transformation once, it goes back to the identity. OK? So here, if you do the twice-- yeah, if you do it twice, it goes back to -- if you do it twice, it goes back to identity. And this is the same. OK? So here is a Z2 transformation.

Also, symmetries can be separated into continuous symmetry or discrete symmetries depending on whether you have continuous parameters or not. But symmetries can also be separated into other different-- there's another classification of the symmetry. It's called the gauge symmetry and the global symmetries.

So there are global symmetries-- corresponding to the transformation parameters are spacetime independent. OK? So both of these are examples of global symmetries because they are just constant. OK? They don't-- a mu don't depend on spacetime. And the lambda don't depend on spacetime. OK.

And you can also have local symmetries, in which case the transformation parameters are spacetime dependent. OK?

So an example of the local transformation which you may remember is the so-called gauge transformation in E&M. So E&M, you can-- yeah, anyway. So later, we will see an example. We will go back to E&M again. And you will see examples of this local symmetry. Any questions? Yes?

AUDIENCE: So if the transformation changes the action but doesn't change the equations of motion, is that still a symmetry?

HONG LIU: Yeah. Yeah, phi deals with the action-- yeah, that's a very good question. So by definition, if it leave the action invariant, we say it's a symmetry. And in general-- actually, almost always, if it leaves the action invariant, actually it also leaves the equation of motion invariant.

Yeah. Yeah, but it's not guaranteed. Yeah. Just purely from a mathematical point of view, it's not guaranteed. But for all physical examples, it's almost always the same. Other questions? OK. Good.

And now there's a very simple-- it's simple but a very deep connection between the symmetry and the conservation laws. So that you may have already learned it in-- yeah, actually I forgot when I learned it myself. So some of may have learned it in high school. Some of you may have learned it in 8.01, et cetera.

Anyway, so there's a connection between the symmetry and the conservation laws. And so any conservation laws can be understood as a consequence of symmetry. So in classical mechanics, you may remember that the time translation-- this leads to energy conservation.

And the spatial translation symmetry then leads to the momentum conservation. And if it's a rotational symmetry, it then leads to the angular momentum conservation. OK?

So this should be something you already know, say, from your classical mechanics. But in classical field theory, this can be generalized. OK? So here, there's the Noether theorem-- so first discovered by Emmy Noether, the Noether theorem. She said any continuous global symmetry leads to conservation laws. OK?

OK. It leads to a conservation. And so no matter what kind of symmetry, OK-- in addition to those things we are familiar with, but no matter what kind of symmetry-- but any time you have a continuous global symmetry in your system, then you have conserved. You have conservation.

OK. So now let me give a proof of this Noether theorem. OK? So before I prove it, do you have any questions regarding its statement? Yes?

AUDIENCE: So what's the definition of-- so you said a continuous symmetry is one that depends continuously on a parameter. But if you wanted to write out, like, equation star-star-star in a way that specializes to continuous symmetry, is there a way to do that-- to show just how to check if something is a continuous--

HONG LIU: Oh, yeah, yeah. You just check whether there's a continuous parameter. You just check whether there is a parameter or not. Yeah. Does this answer your question? Yeah. Normally, you can just see whether there's a parameter. Yeah. [LIGHT CHUCKLE]

If you write down the transformation explicitly, you will be able to just see. If we go over here, there's no transformation parameter. You can just see. And then there, for the Lorentz transformation and translation, you can just see it explicitly. OK.

So an important aspect of a continuous symmetry is that, because you have a continuous parameter-- so you can continuously relate it to identity. So trivial transformation-- you just don't transform at all, right? So in this case, the a mu is equal to 0.

And so in this case, the trivial transformation-- just the lambda mu nu-- is the identity matrix. And then you can imagine a slightly-- a thinning on the infinitesimal rotation or infinitesimal boost. And then that's corresponding to a so-called infinitesimal transformation. OK? So here, you can also just translate a mu a little bit, OK-- so very close to the identity.

And so essentially, for any-- OK? So for any global continuous symmetry, it has an infinitesimal form.

OK? It's just when your transformation parameter is very close to the identity, OK, very close to the 0.

And so for example, for a general transformation like this, if the transformation is close to the identity, then we can always write it in the following form-- phi a goes to phi a prime, which is equal to phi a, and with some parameter-- small parameter epsilon-- and some function-- f_a-- and then, say, phi b, OK, and the derivative of phi b, et cetera. OK?

So this epsilon is the infinitesimal transformation parameter. OK? So if you have a continuous symmetry, there's always, say, an infinitesimal transformation which are close to the identity. So this epsilon corresponding to this case for a mu is very small. Or for the Lorentz transformation case corresponding to rotation angle, it's very small-- or the boost is very small. OK? Yeah. So the epsilon can be any of those parameters.

And then this f, it can be some arbitrary function. OK? So this f can be some arbitrary functions of your fields. OK. Good?

So in this case, yeah, the data phi, OK-- yeah, so the symmetry transformation-- so we'll use the rotation delta-hat phi a. So this symmetry was then given by the epsilon f a OK? So delta-hat here implies an infinitesimal symmetry transformation. OK.

So now let's consider during a variation of S under this delta-hat, OK, since by definition of the symmetry-- because this is a symmetry, that means that delta-hat S should be equal to 0. OK? Because of the variation of the action, it should be invariant under the symmetry.

And so now let's look at what's the consequence of it. So now we can just vary this L. Then the-- so this means that the variation of the L, the Lagrangian density, must be a total derivative, OK, since every transformation is proportional to-- yeah.

So epsilon is a small parameter. So when we do the transformation, we only need to keep track in the order in epsilon. And so the transformation of the L must also be proportional to epsilon. OK?

So since this must be invariant, that means the variation of the L must be a total derivative. So k mu would be just some-- so for some k mu. OK? For some k mu. So there must exist some k mu. And then this k mu can also be 0. In that case, the Lagrangian density is invariant. Yes?

AUDIENCE: Is delta-hat the variation under your transformation?

HONG LIU: Sorry?

AUDIENCE: Is delta-hat the variation under the transformation?

HONG LIU: Delta-hat just denotes such a transformation. It just denotes such a transformation. This is not the general variation. This is very specific. This is a symmetric transformation. And so under the symmetric transformation, your action should be invariant by definition because this is a symmetry.

And that in turn-- because the S is the integration of L, then that means that the transformation of L, the Lagrangian density, must be a total derivative. And here, we keep track of everything in the order in epsilon. And so this must be proportional to epsilon times the total derivative. OK?

And so we must have this structure, yeah, for some k mu, OK, which can be 0, which can be 0. OK? Good?

So now let's look at what's the implications of this equation. OK. So this equation is the constraint imposed by the symmetry. OK? Now let's look at the implications of this constraint.

So now we can just do the variation of L. Now we can just do the variation of L. Yeah. Maybe I will keep that equation. And then I will do this board first.

AUDIENCE: So I have a question. Does this problem--

HONG LIU: Yeah?

AUDIENCE: --show if something is a symmetry of your action for your system, would you directly vary the thing-- like, parameter and then show that your action doesn't change? Or would you do that infinitesimal form and then show that--

HONG LIU: So we want to--

AUDIENCE: Yeah, I don't know. I was just--

HONG LIU: Right. So we want to use the fact that such a transformation is a symmetry to derive what is the constraint on the Lagrangian, OK, what is the constraint on the theory. Say, we assume that somehow the theory has a symmetry. And then we want to see what is the constraint this put on your theory. Yeah, that's right. We are proving the laws of the theorem. Yeah. Good? Other questions?

OK. So that equation is the implication-- is the constraint imposed by the symmetry. And we want to see what this equation tells us. OK? And for this purpose, we just do the variation. So the delta-hat L-- so because L is just a function of phi-- and then we have partial L, partial phi a, delta-hat phi a; and then we have L, partial-partial mu phi a; and then partial mu delta-hat phi a. OK?

So we just did the variation. And now-- OK? And so-- yeah. And now we are going to use the equation of motion. OK? So from the equation of motion-- I think I erased-- yeah, from the equation of motion, then partial L, partial phi a-- this is just equal to partial mu, partial L, partial-partial mu phi a. OK? So we just plug this into here.

So when you plug this into here, now you notice this becomes a total derivative. OK? So you partial L, partial mu phi a, and delta-hat phi a. OK? So you find-- after using the equation of motion, this variation of L have the following form, which is the total derivative.

And this should be equal to the right-hand side. OK? It should be equal to the right-hand side. Yeah. So yeah. And then here we find-- so this should be equal to epsilon partial mu k mu. OK?

So we combine both sides together, plug in that this is equal to epsilon times fa. And then we conclude, OK, partial mu, partial L, partial-partial mu phi a, and fa minus k mu, we see equal to 0. OK? So if we call this J mu and then we have a conservation equation-- partial mu J mu equal to 0. OK? So we have a conservation law.

So remember, conservation law is just to have a vector which satisfies this equation and then-- yeah. So more explicitly-- yeah, so the Jmu is just this combination. OK? Jmu is just this combination. So any questions on this? Yes?

AUDIENCE: Is this partially with the shift in presence because it's the derivative with respect to spacetime here, but if we have a conservation law from the derivative with respect to time equal to 0 but not space?

HONG LIU: Yeah, yeah. I will-- yeah. Yeah, this is the stronger version than your normally-- yeah. So this is the spacetime version of the conservation law from your normally classical mechanics. Yeah. So this is a field theory version of that. Yeah. Yeah, I will elaborate that equation a little bit. Yeah. Yes?

AUDIENCE: So if you only transform the field, not the x and t for the--

HONG LIU: Yeah.

AUDIENCE: --the effect is to eliminate that derivative of your operator for everything? Is that right?

HONG LIU: Say it again?

AUDIENCE: I mean, the reason you include the derivative of the field is because you are including the effect of transforming the spacetime coordinates.

HONG LIU: So here-- yeah, yeah. So here is a general formulation. So this transformation can be general. It doesn't have to be, say, a spacetime transformation-- we said earlier. This is just some abstract symmetry, just some arbitrary transformations.

AUDIENCE: Yeah. I mean, if you only go through the symmetries that have changed in the field but not the symmetries that haven't?

HONG LIU: Yeah, because spacetime variable is a dummy variable. In the action, you can always get rid of that. The x change-- x is just a dummy variable in your action because you integrate over x. Yeah, yeah.

AUDIENCE: So is it possible to interpret higher order derivatives not including that pathway?

HONG LIU: Yeah, yeah, yeah. This definitely include the-- normally-- yeah, you can in principle have as many derivatives as you want. But normally, if we have an action which only contains first derivatives, then your symmetry transformation will only involve first derivatives.

AUDIENCE: So, though, we're just not discussing the case of transforming-- I don't know. We're--

HONG LIU: Yeah, yeah, yeah. In principle, you can have it. In principle, you can have it. Yeah, I'm just writing this for simplicity. Yeah. Yeah, good. Maybe you can ask after the class, yeah, if it's not clear. Other questions? OK. Good.

So let me elaborate a little bit on this equation. So this equation you may have seen in E&M, which you often call the continuity equation. If you write this equation separately, it has the following form-- partial 0 J0 plus partial I, or the divergence of the spatial component. So J mu-- if we write it in terms of the-- OK, explicitly, then you have this form.

So this is so-called the continuity equation-- just the time variation of the density. OK? The 0 component that you consider as some kind of density-- it's the same as the divergence of the current. OK? And in particular, you can define a charge, which is the total spatial integration of J0. OK?

So if you integrate both sides over the total volume-- and then this just becomes partial 0 Q. And this term becomes a total derivative. And then you can convert it into a boundary term using the Gauss's law. And then, normally, the current vanishes at infinity. And then you just have the charge conservation. You just have charge conversion. OK. And yeah.

So this is the field-- so normally in classical mechanics, you just have this equation. OK? And this is the field theory version of your conservation. Yes?

AUDIENCE: So is there a way to maybe to know what it is and how we got it?

HONG LIU: Sorry?

AUDIENCE: Is there a way to calculate what J represents?

HONG LIU: J represents?

AUDIENCE: Yeah, what it represents?

HONG LIU: Yeah, yeah, yeah. If I give you a specific-- yeah, you will do it in your Pset. So if I give you a specific Lagrangian-- if you start a specific transformation, and then you will be able to find all those quantities-- you will be able to find the k explicitly. And you will be able to find all those quantities explicitly. Then you can find the J. Other questions? Yes?

AUDIENCE: Minus delta L of 0 implies that delta L needs to be over here.

HONG LIU: Right. It's because if we have here-- so here, if we have delta s equal to 0-- and then you can just put the delta in. And then this quantity that is integrated over that-- if this is 0, then this has to be a total derivative Other questions? Yes?

AUDIENCE: So what happens if it's not a global symmetry? Where does this argument break?

HONG LIU: Right, yeah. Good. This is a very good question. So yeah, I will not have time to go into here. Let me just very quickly mention. So this is called the first Noether theorem-- and when epsilon is spacetime independent.

And so when epsilon is not spacetime dependent, then the story is a little bit more complicated. So the epsilon will be inside this total derivative. You cannot take it out, et cetera. And then the story changes a little bit.

And then there's something called the second Noether theorem. And actually, in that case, when you have local symmetries when epsilon depends on spacetime, instead of finding conservation laws, you find that your equation of motion-- not all-- you find some parts of the equation of motion are redundant. Yeah. Yeah. And if you-- it's not difficult. But actually, in principle, I can put it in the Pset problem, if people really want to see it. Yes?

AUDIENCE: So this is something to think about the transformation, the vicinity of the identity. What happens to the information of the transformation outside of--

HONG LIU: Good, good, good. So this is the beauty of physics, which you may already have seen in quantum mechanics. So whenever we see a symmetry, no matter how complicated that symmetry is-- say, some continuous symmetry-- it's enough just to understand that symmetry near the identity, near to the infinitesimal transformation.

It's because any finite transformation you can build up from just adding up the infinitesimal transformations. So once you know the infinitesimal transformation, actually, essentially up to some global or topological structure, essentially it determines the full finite transformation. Yeah?

AUDIENCE: So we can define transformation by taking the derivative of things that are different that obviously--

HONG LIU: No, no. It's always a symmetry, right? Because each step is a cemetery. And so you add them up. You add them up still a symmetry.

AUDIENCE: OK. But a sequence of different symmetries, for example?

HONG LIU: Oh, if you do different-- yeah, they're still a symmetry. Yeah, because by definition if something transforms that's invariant into another transformation, it's still invariant. Good? Other questions? Yes?

AUDIENCE: For discrete symmetry, where we don't have this epsilon, can we do something similar?

HONG LIU: Sorry?

AUDIENCE: For discrete symmetries--

HONG LIU: For discrete symmetry?

AUDIENCE: --do we have to do this analysis? Or can we can do something similar?

HONG LIU: Yeah. In general for discrete symmetries, you don't have such a conservation law. You don't have such a conservation law. But you can still sometimes define some discrete quantum number which is conserved, like the parity which-- in the parity case. But you don't have a current. OK? So here, J mu is called the conserved current. But you won't have a current. Good? Good.

So this concludes our very quick discussion of the important features of classical field theory. And with this preparation, now we can start talking about or start thinking about going to quantum mechanics, talking about the quantum fields. OK?

And so here is a good place to pause a little bit from what we already said about the classical field theory to anticipate a little bit about the quantum fields theory. OK? So here, I will restate our goal for quantum field theory, which I said a little bit in the last lecture. OK?

So now we have seen the classical fields. And our goal is to understand the quantum version of this story. OK? So classical field theory is classical mechanics with infinite number of degrees of freedom. And now we want to quantize this infinite number of degrees of freedom. And now this becomes quantum field theory. OK? And now we want to understand the quantum dynamics of such kind of phi fields. OK.

And for example, an example is the E&M. We have electric and magnetic fields. OK? And you know the Maxwell equations. You can solve the Maxwell equations. And so this defines the classical field theory. And then we will tell you how to actually quantize such a system to understand the electric field and the magnetic fields are quantum mechanical. OK.

Good. And that actually would be our goal. OK? So we will start with the simplest field theory-- just the scalar field theory. And then we will go to the Dirac theory, which describes the electron. And then, eventually, we will go to the theory which describes the electromagnetic field. So this is called the QED, OK-- quantum electrodynamics. So that will be the endpoint of this course.

OK. So now let's say a little bit how we go through this classical field theory to quantum field theory, OK-- so just some general remarks. So before doing that, let's think a little bit how we go from classical mechanics, with a small number of degrees of freedom-- and how you go to quantum mechanics. OK?

So let's recall. So in classical mechanics-- so let's just consider the simplest case, if you just have a single degree of freedom-- say x(t), OK-- just a one-dimensional particle whose motion just describes the x(t).

And then, of course, the goal of classical mechanics is just to solve the equation of motion over x(t). OK? So if I write down the equation for x(t) and I solve it, then I'm done. OK? I'm done. I solved the classical mechanics.

So when we go to quantum mechanics, what do you do? So this x(t) then becomes an operator. OK? So it becomes an operator. So the classical dynamical variable now becomes a quantum operator. And the equation of motion of x(t), if you remember-- and do you remember what it becomes?

Yeah, exactly. It becomes the Heisenberg equation. So it becomes the Heisenberg equation for this operator x(t) now. OK? Yeah.

So in quantum mechanics, there are normally two ways to describe it. So first is the Schrodinger picture. In this case, you have a wave function, which is the function of x and t. And then you have an operator, which is x. OK? And then, of course, you also have conjugate momentum, et cetera, which is defined to be x dot. OK?

And so the wave function is a function of the eigenvalues of-- so x here should be understood as eigenvalues of this operator x-hat. OK? It's the eigenvalue of x-hat. So in the Schrodinger picture, you solve the evolution for psi. You solve the equation for psi. And then once you find psi, then you can calculate any quantities you want. OK? You can calculate any expectation values, any amplitude, et cetera.

But the second way of approaching it is the Heisenberg picture. In this case, you-- the dynamical quantity is your operator. Now you're assuming the operator depend on time. So in the Schrodinger picture, your x is just some constant operator that does not depend on time.

But in the Heisenberg picture, now your operator now becomes time-dependent. OK? And the equation of motion-- and then you solve the Heisenberg equation. OK? And then the state is invariant. OK? The state does not evolve with time. OK.

So you solve for the Heisenberg equation for those operators. And then once you solve the Heisenberg equation, and then you can evaluate, again, your expectation value in any state you are interested in, et cetera. OK. So any questions on this? So this is a very quick review of what you did to go from classical mechanics to quantum mechanics. OK.

So now in field theory, OK, we do the similar thing. So similarly, in field theory, we have classical fields. OK? We have classical fields. And then when you go to quantum, this become quantum operators. OK? So this is now quantum operators.

Remember, in field theory, this x is always just labels of the space points. OK? It's not a dynamical variable. OK? The notation is a little bit-- yeah. Here, in classical mechanics, this is a dynamical variable. But in field theory, this is just labels. OK? So they're not dynamical. The dynamical variable are phi itself.

So phi becomes an operator. OK? And then now the classical equation of motion for phi, which we derived-- and then become the Heisenberg equation, OK, for phi-hat.

So again, here, you can do two pictures. You can do the Schrodinger picture, or you can do a Heisenberg picture. So for QFT, the Schrodinger picture-- OK. And you look at the wave functions of your dynamical variables, the eigenvalue of dynamical variables.

So what is the generalization of this psi xt? So remember, here, psi is a function of eigenvalue of your dynamical variables. OK? So now, if we push this analog further, when you go to quantum field theory in the Schrodinger picture, now the wave function should be a function of phi (x) and the t. OK?

So this phi x are eigenvalues of phi-hat x. OK? So in the Schrodinger picture, phi-hat does not evolve with time. And so you have the wave function, which is the function-- now become a functional-- because this itself is a function of space. OK? And yeah.

And then you solve the Schrodinger equation for this. And then you have the operator. And then you have the operator. Yeah. And so you have-- and then the dynamical-- and then you have operator phi-hat x. OK? So that's what you do in the Schrodinger picture, in the Schrodinger picture.

And in the Heisenberg picture, you forget about the wave function. OK? You look at the evolution of operators.

So the Heisenberg for the field theory-- your Heisenberg picture for the field theory is that you look at the evolution of x and t. So now this just obeys the Heisenberg equations, which is just the quantum version of the classical equation for phi. So you look at the evolution of this. And then the state does not change with time. OK? The state does not change with time. Good. Any questions on this? Yes?

AUDIENCE: When it comes to-- so we have the quantities--

HONG LIU: Yeah.

AUDIENCE: --what happens to time in the Schrodinger picture?

HONG LIU: So in the Schrodinger picture-- so remember, the x is a label for phi. But time is the evolution. But in the Schrodinger picture, the operators don't evolve. And so there's no time here.

So we don't have time here. And we just have the analog of the x here in the Schrodinger picture for classical-- yeah, for quantum mechanics. And then the time dependence is in your wave function. So the wave function is a function of possible values, eigenvalues of this operator phi. Yeah. OK?

And then the Heisenberg picture-- again, then you just focus on the operator equations. And once you solve the operator equations, and then you can calculate the expectation values in any state you want. OK.

So now you can already maybe see a difference a little bit. So if you do the Schrodinger picture, you have to deal with this beast, OK, which is the wave functional, though, of all possible values of some function in space. OK? And if you have multiple fields, then this is a hugely complicated thing. And you need to write down the Schrodinger equation for it, et cetera. OK?

But here-- but in the Heisenberg picture, we just solve the analog of the classical equation of motion, which we have already written down. And you just now interpret it as a quantum operator equation. So which one do you think is simpler?

So that's why in quantum field theory we almost always use Heisenberg picture. OK? We always use Heisenberg picture. And we don't even-- rarely think about the wave function, even though sometimes this can be useful in some problems. But for most of the time, this is much easier. OK?

So that's what we are going to do. OK? And so in quantum field theory, we will just use the Heisenberg picture almost all the time. OK? From now on, I will not talk about the Schrodinger picture. OK.

Good. So it's very important you remind yourself about quantum mechanics in the Heisenberg picture, OK-- And because most of your quantum mechanical classes before maybe is all in the Schrodinger picture, solving the Schrodinger equation, et cetera. But now you have to change the perspective to think of everything in terms of the Heisenberg picture. And then that will make you-- quantum field theory much easier. OK?

Good. Any questions on this? Yes?

AUDIENCE: So in the Heisenberg picture, if we're not concerned about the wave function and all, then how do we determine, for example, the probability or the expectation value of where x is at some point from the momentum?

HONG LIU: Good, good, good. That's a very good question. So this is actually also related to the kind of questions we want to solve in quantum field theory. So in quantum field theory, we often work with the vacuum state. So for example, here-- yeah. So when you can see that the QED-- most of the time in the quantum electric or magnetic field, they're in the vacuum state.

And so we just consider the psi in the vacuum state. And so we don't have to consider-- yeah. Right. Yeah. And so normally in quantum field theory, there are preferred states we are interested in. But that's a lot of the reason that the Heisenberg picture is convenient. Yeah. You don't have to consider the general state, in general. Yes?

AUDIENCE: So what is the physical meaning of the wave function as it goes beyond the--

HONG LIU: Hmm? Sorry?

AUDIENCE: What is the physical meaning of psi here?

HONG LIU: Oh, this is just some state you are interested in. Still, you have a Hilbert space. But now you just don't-- when you study the evolution, you only evolve the operator. You don't evolve states. Yeah, so that's the difference between the Schrodinger and the Heisenberg picture.

AUDIENCE: But for example, in quantum mechanics, we get the expectation values of some operators with respect to the state. And we get, like, measurements-- the results and stuff. So what about here?

HONG LIU: Yeah, it's the same thing. Yeah, same thing. Yeah. At this level, there's no difference between quantum field theory and quantum mechanics. Yeah, just think about quantum mechanics in terms of Heisenberg picture. Translate everything you learned about the Schrodinger evolution, et cetera, in terms of Heisenberg picture. Yes?

AUDIENCE: And now the open state is conventional.

HONG LIU: Hmm?

AUDIENCE: The open state is--

HONG LIU: Yeah, that's right. That's right. Yeah. Yes?

AUDIENCE: Am I right that the state we're interested in is generally the vacuum state. And what sorts of measurements--

HONG LIU: Yeah. So it's-- again, so-- yeah, it's a very good question. What I should have said is that the states that we are interested in are states which are close to the vacuum state. So you excite the vacuum a little bit.

And yeah, it's not-- yeah, of course, in the vacuum, you don't have anything. And we are since close to the vacuum state. Yeah. And later, when we discuss things, you will see. Yeah. Good. Good. OK.

So before going into quantum field theory, let's also make some remarks, have a short discussion on relativistic quantum mechanics. OK? So naively, if you have special relativity plus quantum mechanics-- so if you want to generalized-- so most of the quantum mechanics you learned is the non-relativistic quantum mechanics. OK?

But now if you want to incorporate special-- we want to combine the special relativity with quantum mechanics. And then what should you get? What do you think you should get?

AUDIENCE: [INAUDIBLE]

HONG LIU: Hmm?

AUDIENCE: [INAUDIBLE]

HONG LIU: I expect-- yeah, just whatever come to your mind.

AUDIENCE: [INAUDIBLE]

HONG LIU: Hmm?

AUDIENCE: [INAUDIBLE]

AUDIENCE: [INAUDIBLE]

[LAUGHTER]

HONG LIU: That's the right answer. That's the correct answer. But I was hoping some of you will say it's relativistic quantum mechanics.

[LAUGHTER]

OK? So naively, when you combine these two-- in particular, if you read some old quantum mechanics books, they do discuss relativistic quantum mechanics. OK? So naively, that's what you get, OK, when you combine these two. And you say, oh, we just get the relativistic quantum mechanics. OK?

But that's actually not a correct statement. Actually, strictly speaking, the reason now you don't actually learn much about the relativistic quantum mechanics is because, strictly speaking, relativistic quantum mechanics does not exist. OK? And whenever-- if you want to combine quantum mechanics with special relativity, actually, you get the quantum field theory.

So the quantum field theory is actually forced on us if you want to unify special relativity and quantum mechanics. OK-- and even if you don't want to talk about the fields. OK? Even if you don't want to talk about fields, if you just want to talk about particles, still, if you want to unify these two, then it actually automatically leads to quantum field theory. So now let me just explain why this is the case, OK, so that you have some better appreciation of the quantum field theory. Good?

So let's try to apply what you did for non-relativistic quantum mechanics, try to generalize it to derive some relativistic quantum mechanics. OK? Suppose you are the people in 1926. OK? Quantum mechanics just was discovered. And you say, oh, people have understood the non-relativistic quantum mechanics. Now let's generalize to special relativity.

OK. So in non-relativistic quantum mechanics, how do we derive the Schrodinger equation? The way we do this is we start with the dispersion relation, OK, for a non-relativistic particle.

And then we say this-- take E to I partial t, OK, p-- so say, if you kind of have a vector become-- yeah, so don't let me forget about h-bar. Just p then becomes spatial derivatives. And then this equation just become the Schrodinger equation for the free particles.

OK. So this becomes the Schrodinger equation for free non-relativistic particles. And then you can add the potentials, et cetera. OK. So that's how you derive your Schrodinger equation for non-relativistic quantum mechanics.

But if you are someone, say, in the early days of quantum mechanics, say, now let's try to generalize it to special relativity. And then it's easy to do then in the relativistic case. Then you have-- we just start with the relativistic dispersion relation. OK?

You say, let's do the same thing. OK? Let's do the same thing. So this becomes I partial t. And this p becomes this one. And then we can just write this equation. OK?

So now let's combine with these derivatives together-- or put all this on the same side. And then what you find is partial mu, partial mu. OK? That's what you get.

So does this equation look familiar? OK? So this is the simplest scalar field theory equation of motion we have written down. But here, the interpretation is very different. OK? So earlier, we wrote down this equation. We say this is Klein-Gordon equation.

So this is the Klein-Gordon equation. OK? So this is the equation of motion for the simplest free-field scalar field theory we wrote down before. But here, the interpretation is very different.

So remember, the psi is not a field in quantum mechanics. It's a wave function for a single particle sitting-- so the psi is the amplitude for a particle at spatial location x at time t. OK? So this is-- and it's square gives its probability. OK?

And this does not describe a field. OK? This is a wave function of a single particle, OK, even though they have the same equation as the field theory we wrote down earlier.

And so that's what Klein-Gordon did. OK? So the Klein-Gordon-- we try to generalize non-relativistic quantum mechanics to relativistic quantum mechanics. And then they wrote down this equation. They say, ah, now we are immortal. OK?

[LAUGHTER]

Because we wrote down the first equation for relativistic quantum mechanics. And then soon realized, actually, this equation-- if you want to interpret it as the right equation for the wave function, we have actually lots of problems. There are various problems. Yeah, by saying "lots of"-- maybe it's a little bit of an exaggeration there. There are various problems.

So yeah. So some-- yeah, let me call this equation-- I think I've used up my stars. I think to have a four star is a little bit too much. Let's just call this equation "1" for this section.

So some immediate difficulties of interpreting 1 as a wave equation, OK, for a relativistic particle-- yeah, let me just save time. OK?

So this is the wave equation for a non-relativistic particle. If you want to generalize to relativistic quantum mechanics, then you would interpret this as a wave equation for a single relativistic particle. OK.

So first is that-- so if you remember-- so in this equation, OK, for-- if you go back to your early days of non-relativistic quantum mechanics-- so you remember, from this equation, you can derive an equation for the conserved probability, OK, a conserved probability.

And that equation tells you that psi-squared should have the interpretation of the probability. OK? So-- but for that equation, you can show there's no quantity that can be-- that no quantity that can be used as probability density. OK?

So probability density by definition should satisfy two conditions. So the first condition is that it's non-negative. And the second condition is that it should be conserved. OK? The probability should conserve. Otherwise, you violate the-- yeah.

So you can show that this equation does not allow. This equation allows such a quantity, but this equation does not allow. OK? And I will not go through that myself. I think that will be in your Pset 2. OK? That will be in your Pset 2. You will show it yourself.

And the second difficulty is that if you square that-- if you find the energy-- if you take the square root, then the energy in principle can be, say, plus-minus. You have two solutions. OK? E equal to plus-minus this quantity.

So in contrast to that equation, there's only positive energy. OK? In non-relativistic, you just have positive energy. So classically, you can just say, let's throw away the second branch. OK? We can just throw away the second branch phi-hat, classically. But quantum mechanically-- OK? So classically, just ignore the second branch, the negative branch.

OK? But quantum mechanically, this is not possible because you have the equation there. Then you automatically have the negative solutions. OK? So quantum mechanically-- and then you have a dispersion relation like this. So that dispersion relation has the following form-- is phi E as a function of p.

And so you have a positive branch. And then you have a negative branch. And then you remember, in quantum mechanics, you have energy levels. And you have energy levels. And such particles-- the particles here then have a higher energy than particles here. OK?

And so here, the energy level is higher than the energy level here. And there's always a non-zero probability quantum mechanically for a particle to go from some higher energy level to the lower energy level, OK, just like in the hydrogen atom. If you excite it, they always go to the lower energy level.

So then quantum mechanically, such a thing cannot be avoided. OK? So you cannot just throw this branch away. OK. And so this will lead to instability because the energy can be infinitely negative. OK? It can be infinitely negative. And then all your particles will all go to infinitely negative energies. And then your system will be in big problem. OK.

So these are the two-- are the most prominent problems. And then people tried many different ways to try to avoid them, et cetera, including some very ingenious solutions, et cetera. Though, we will not go into them.

But let me just mention there's actually even-- if you can address those problems, still, the relativistic quantum mechanics will not make sense for a very fundamental reason. So these are more like superficial reasons why that equation does not quite work. OK? But there's actually a more fundamental reason why relativistic quantum mechanics even as a concept does not make sense. OK?

So by definition, if you want to interpret this kind of thing as a wave function-- OK? So what's an interpretation of the wave function, which we already said? So this describes the amplitude for a single particle at some point, OK-- at some point x at time t.

But now, if you have two particles, what do you do? You introduce the location for particle 1, and the particle 2, and the t. OK? So this describes two particles. And if you want to describe three particles, then you have to introduce more x. OK? Remember, that's what you did in non-relativistic quantum mechanics.

So in non-relativistic quantum mechanics, this makes sense, OK, and just because there's no mixing between the different branches-- say, single particle, two particles. Because a particle cannot be created and destroyed in a non-relativistic system. But in a relativistic system, you can always create particles. OK?

If you have enough energy, you can create new particles. A pair create electrons. OK? It happens in the accelerator all the time. Or electrons-- they can annihilate into photons. OK?

So the particles are not conserved, OK, are not conserved. So this kind of wave function concept don't even make sense. So if you want to describe-- so that means, whenever you have special relativity and quantum mechanics together, you must have a framework which can describe arbitrary number of particles at the same time. OK? Because particle numbers can change all the time, OK, because of annihilation and the creation effect.

And that cannot be achieved by this kind of concept. It cannot be achieved by wave function. It turns out miraculously that it can be achieved by field theory. It turns out that field theory, once you quantize it, automatically gives you a framework to describe arbitrary number of particles, OK, in the unified manner, OK-- single particle, two particles, arbitrary number of particles.

And so this is one of the magic of field theory. So that's why the field theory plays such an important role in particle physics, is because the excitations of fields automatically provides the mechanism to describe arbitrary number of particles. OK. So we will stop here today.