[SQUEAKING]

[RUSTLING]

[CLICKING]

[SIDE CONVERSATIONS]

PROFESSOR: OK, let us start. So, last time, we discussed how to calculate such a correlation function, say Gn, in a single particle theory, so using path integral. So the goal is to calculate this time-ordered product correlation function, the vacuum correlation function, of this time-ordered product in this theory.

So, last time, we described how to do this using path integral. And we derive the beautiful formula. So the formula is given by the following. It's a Gn is given by the ratio of two path integrals DX(t). So if we call this thing to be x. So the x and the exponential i S xt. And then, the DX(t), just the pure path integral.

So, here, I didn't write to the upper limit and lower limit. So it should be understood that the boundary condition for both path integral is that the x-- So it should be from minus infinity-- the time range should be from minus infinity to plus infinity. And then we can choose the value of x at both ends just to be zero.

And, yeah, and the slight subtlety is that, when you evaluate S, so there should be the epsilon parameter because we-- in deriving the path integral we have, say we need to give the Hamiltonian a slightly say epsilon part. So that will also affect your action. So your action will also have a small epsilon dependence.

And so, at the end of the day, after you have done this calculation, and then you said epsilon goes to 0. So the epsilon goes to 0 at the end of here.

So when we write the epsilon, it's always should be assumed that this is a positive number. This is infinitesimal positive number. Good. Any questions on this?

So, now, in principle, with this formula, then we can now calculate this quantity. We can calculate this quantity. And we can calculate this n-point function.

So, but, in practice, it's actually more convenient, rather than to calculate the Gn, because we often need to know the such correlation function for different n. We often need to know such correlation function for different n. So not only say sometimes we are interested in n equal to 2, sometimes interested in n equal to 3, 4, et cetera.

And then, there is a nice trick to-- you can try to calculate, then say, in general, is to use this technique called generating functional, which we started talking about at the end of last lecture.

And so, the basic idea of the generating functional can be easily understood by consider just this one-dimensional example. Say, if you are interested in doing an integral like this, xn. And, if you are interested in this integral for different value of integer n, then it's more convenient to consider such an integral, Za given by--

So the reason I put the i here is just, yeah, I didn't specify the range of x. And if x is from minus infinity to plus infinity, it's not easy to-- yeah. Yeah, put i here, just so that this integral can be defined. Depending on circumstances, you don't have to put the i. Say if x is from 0 to infinity, then I can just put, say, minus lambda a, with lambda to be a positive number. And then, that's fine.

So the benefit of considering this integral is that-- oh, no, no, no, not the-- sorry, I should-- x, yeah, xa. Yeah. So the benefit of doing this integral is that if you notice, say, if you take a derivative with respect to a, and then, that will bring down a factor of x.

So it's a-- so if you take a derivative with a, you bring down a factor of x, and you take the derivative twice with a, then you bring down a factor of x square. And if you do it n times, and then you bring down a factor of x to the power n.

So, essentially, this Zn then can be written as partial n Za, then partial a, you derivative n times. And then you set a equal to 0 because, in the end, we want-- in this integral, there's no exponential piece. And you set that equal to 0, then you get rid of the exponential piece. Then you have factor of xn. So there's still-- yeah. So I still need to put i n here.

So if you know how to compute the Za, and then you only need to take derivatives to do Zn. So taking derivatives is much easier than doing integrals. So, in other words, we can also write it, expand Za in terms of power series in a.

And then, the Zn would be the coefficient. So Zn would be the coefficient for a to the power n. Yeah. Good? Yeah, so we call this Za the generating function.

So now we can use the similar idea to generalize to this case. And, now, here, this is-- here it's just one-dimensional integral. Here we have a functional integral. So now this is the function. So, essentially, you just generalize this generating function into a generating functional. And so, we can consider the following object.

So in order to compute this, then we can consider the analog of this a is we consider object called J. J now will depend on t. Yeah, J is a function. J is a function because x now becomes a function. And so, introduce a path integral like this.

So, from now on, I will not-- when I don't write the range, then it's always-- you should always keep this in mind-- the range will always from minus infinity to plus infinity with the boundary value to be 0.

And then, we can just say, I have standard xt. Then I can add the analog of this. So, remember, again, you think t is just an index. And then you just, essentially, imagine if you have multiple x. And then you just sum over them. So sum over them, in this case, just corresponding to an integral. And so, we just have a piece like this.

So this would be the generalization of that equation. And so, integration over t just can be imagined as a sum. Imagine if you have multiple x and multiple a, and then you need to sum over them. And then, yeah, so this is just a generalization of that. Do you have any questions on this?

So now, similarly, now if we take a functional derivative with respect to Jt, then we can bring down a factor of xt. So, more explicitly, so let me just remind you, that the rule of doing functional derivative. So if you have a function, delta Jt prime, with respect to the derivative of Jt, and then that just gives you a delta function. That just give you a delta function.

So now let's look at-- with this rule, then let's just look at the delta, delta Jt on the ZJ. So and then, yeah. So, now, since this is Jt, let's just, for convenience, let's just put this to be t prime. Let's put it to t prime.

And then, when you take derivative with respect to Jt, and then you directly take derivative with this guy. And then, when you take the derivative, the delta function will get rid of this integral. And then you just have xt. And then you will just get i DX(t) exponential i S. And, yeah, sorry, you will get DX(t). And then you have a factor of xt.

So let me just write Dx-- again, to save, you just add Dx. And you have xt from taking derivative of Jt. And then exponential i S plus i Jx. So let me just save some effort. Yeah, you should keep in mind that this Dt is, actually, I'm just using a simplified notation. Good. Any questions on this?

So, now, if I also introduce Z0, defined to be ZJ equal to 0. So this is just the original integral. So Z0 is essentially the downstairs. So Z0 is just the Dx exponential i S. So when J equal to 0, and then we just have this. So, essentially, this is the downstairs here. So we also-- let's also introduce this definition.

And then we then find-- so, by comparing the two, then we immediately conclude that the one point function of x hat t0 is just given by 1 over i, because this i there, and 1 over Z0, then delta J-- delta ZJ, delta Jt. And then you set Jt equal to 0.

So, similarly, here, if you set a equal to 0, so you take derivative, and then you bring down a factor of xt. And then you set J equal to 0. And then you just have that integral with xt there. And then you get this one point function.

So, now, you can just immediately generalize. So this n-point function, Gn then can be written as, you just take one-- now you have take n derivatives. Now you have a factor of 1 over i to the power n. Again, you divide it by Z0 because you always need to divide by this piece. And then you just take Z derivative n times.

So you take Jt1, Jtn. So this time variable should match with the time variable in the original definition. And then, after you take the derivative, you set J equal to 0. So, and then, this gives you the n-point function.

If you know how to compute this ZJ, and, again, you only need to do derivatives. And then it's much simpler. Then you only need to do the path integral once, and then you just doing the derivatives. Any questions on this?

So, now, if you keep in mind of this, we can also rewrite this expression as the following. We can also rewrite-- alternatively, we can also write ZJ divided by Z0 as following 0 and the time ordered exponential i, dt, Jt, xt, 0.

So, as I mentioned before, that in that formula, in principle, x can be anything. x can be anything. So, now, in this path integral, you can just imagine you separate this term-- the sum of the two exponential, you can just write it as a product.

And then you have the exponential i S, and the exponential i, this piece. And then we treat that piece to be x. And then, that gives you this formula. If you treat that piece to be x, it give you this formula.

Then you ask, what is the meaning when we have a time ordering of this exponential? So the meaning is that, imagine you expand this in power series. Imagine expand in power series of x. And then you just order each term in the power series. You can pick, because now each term is a polynomial, you can time-order them. You can time-order them.

So, now, this object-- so I can also write this object. So when you expand this object, and then you-- the first time is just 1. And then, the next term just Dt, Jt, then the one-point function. And then, under the n'th term, yeah, et cetera.

So you can, when you expand it, you can just write from n equal to infinity, 0 to infinity, i to the power n, n factorial. Now you have n integrals, t1, tn. Then you have times, you have Gn, t1, tn, and the Jt1, Jtn.

So this is the typical-- the n'th term is like that. So when you expand this to n'th power, you group all the x together. Because J just C number-- integration and the J are C numbers. You can take them out. And then this part is just the x between the zeros and then you have this factor of J. Yes.

AUDIENCE: Where did the n factorial come from?

PROFESSOR: Oh, just when you expand the exponential, there's always n factorial.

AUDIENCE: Oh, right, sorry.

PROFESSOR: Yeah. Good. Any questions on this? Yes.

AUDIENCE: When you separate out the exponentials into the e to the i S and then the i integral.

PROFESSOR: Yeah.

AUDIENCE: Is that S, and the other term, like operator. So you'd have to have another contribution from the commutators?

PROFESSOR: Sorry, say it again?

AUDIENCE: So, S, in that case, it functions of an an operator. And then the other one is also like an operator. So wouldn't that introduce commentators?

PROFESSOR: No, no, no. Because, in the path integral, they're just ordinary functions. Right?

AUDIENCE: Oh.

PROFESSOR: In the path integrals, they're ordinary functions. So, in the path integral, they're always just ordinary functions. But then, when we rewrite them in terms of the operator form and then so-- the left-hand side, So the x, they're just ordinary functions. But in the right-hand side, I'm now writing it in terms of the operator form. And now, indeed, now the ordinary matters,

Now the ordering matters. Yeah. So I'm just using this form-- yeah, so, in this formula, the right-hand side is just the ordinary functions. But the left-hand side involving some operator sandwiched between the ground state. Yeah. Other questions? Yes.

AUDIENCE: So in that formula in the middle, where Gn is expressed as the function derivative of Z--

PROFESSOR: Yeah.

AUDIENCE: Does it matter what order I take the derivatives?

PROFESSOR: No, that doesn't matter. Because, again, J is ordinary functions. Yes. Yeah, because this is just the path integral. J is just ordinary functions. This is just some functionals of J. You can just take arbitrary derivatives.

Also, you notice, on the left-hand side, so Gn is a function of t1 and tn. So Gn, as a function of t1 and tn is actually completely symmetric. Because, under this time ordering, it doesn't matter how you order them. Because they're just ordered by time ordering anyway.

So it doesn't matter how I write the ordering here. So Gn is a symmetric function of t1 and tn. So you can see it here. So, here, because all the derivative commute, and then this is a symmetric function of t1 and tn. Yeah. Other questions? Good? OK.

So, in the future, we often just computing this object. We just often computing this object. And then, that will tell us-- then that will give us the generating functional of the correlation functions. Then we can just obtain correlation functions by taking derivatives.

So now let's look at the explicit example to illustrate how this works. So let's just consider simple example, a harmonic oscillator. So, almost always, a harmonic oscillator is a good example. Good?

So, in this case, so, again, we look at this object. But now S is-- so we always now take from minus infinity to plus infinity. We have dt. Then you have 1/2. So let's take this to be my Lagrangian.

So let me call this omega 0. So this is considered the essential harmonic oscillator. So I take the m equal to 1. And I consider this Jt. Yeah. We are interested in computing this object.

So, also, I should mention, in practice, so you always interested in xt. Say, when you calculate n-point functions, even though the value of t-- say, suppose you want to calculate the Gn. So Gn, you have n values of t.

And so, outside that n values of t, you can just take the J to be-- yeah. So we can always take J to go to 0 at the plus minus infinity. Yeah, so that helps your integral to converge. Yeah, this is just a side remark. Good?

So, now, first, to compute this object or this object, we need to first understand what is this S epsilon. We need to understand what is this s epsilon. So, remember, so, for the harmonic oscillator, the Hamiltonian is p squared divided by 2m plus 1/2 omega 0 squared x squared.

And so, here is the Lagrangian, and this is the Hamiltonian. They are related by the Legendre transform. So now, in order to do this H, go to H minus i epsilon. And then, now this become p squared divided by 2m 1 minus i epsilon, then plus 1/2 omega 0 squared, x squared, 1 minus i epsilon. So you multiply the both term by 1 minus i epsilon.

And then, to obtain what is the corresponding S for this, you just do a lot of Legendre transform back into a Lagrangian. So then you find that the L epsilon, which you do corresponding to the Legendre transform of this-- yeah, so this is a trivial exercise, which we can do. Then you find that this give, gets 1/2 x squared 1 plus i epsilon.

And then, yeah, so, this part, essentially, does not change, 1/2 omega 0 squared x squared 1 minus i epsilon. So that's what you get. Essentially, when you invert it, when you do the Legendre transform to go to p, to x, somehow this becomes from 1 minus i epsilon become 1 plus i epsilon. Yeah, you can easily check yourself because you did do an inversion.

And so, now let's write this into a more convenient form. So to write it in a more convenient form, we always-- remember, we always treated the x as a two x's sandwiched by some differential operator. So we can do integration by parts.

So we can do integration by part to write it as minus 1/2 x partial t squared plus omega 0 squared. Then minus i epsilon omega squared plus i epsilon partial t square x. And then plus total derivative.

So the total derivative always vanished because we always-- yeah. Just, we always impose boundary conditions. So that at t equal to plus minus infinity, they go to 0.

And so, now let's look at this object. So let's look at this object, the epsilon dependence. So omega squared is just a positive number. Multiply epsilon, it's still a positive number. And epsilon infinitesimal. So we can just still call it i epsilon. So, now, the partial t square acting on epsilon. So partial t square is a negative definite operator.

Because, remember, whenever you do a Fourier transform on x, so partial t-- a single factor gives you i omega. Then, if you have a partial t square, then give you minus omega squared. So partial t square is a negative definite operator.

So that means that this is also a negative number times i epsilon. And then, that means we can just write the whole thing just as 1/2 x partial t square plus omega 0 square then minus i epsilon x.

Good? Is this clear? Yeah, because just anything, any positives in multiplying epsilon still give you epsilon. Because it's just a small number. It doesn't matter.

So now we can write this S xt, now we can write the S epsilon xt then in the following form in the minus 1/2. Let me write as dt, dt prime as we wrote before. Then xt Kt, t prime and xt prime. And then, Kt, t prime is just given by delta t minus t prime, say partial t prime squared minus omega 0 squared minus epsilon, or plus, yeah.

So, again, we just introduce a lot of t prime. And then I introduce a delta function. And now we have a matrix form. Now, again, this action has a matrix structure. So now, this S depend on epsilon. Now S depend on epsilon. Good? OK? Good?

So now we can evaluate-- now we can now ready to evaluate this path integral. Now we are ready to evaluate the path integral. So let's first look at Z0. And let's first look at Z0. Z0 is just the Gaussian integral we already said before. So the Z0 is the Gaussian integral. So I will be schematic. Yeah, this x dot K dot x.

So this is a shorthand notation to denote this two integral. I think it's positive i. Oh, yeah. Yeah, minus sign. I have a minus sign here. So it's minus i. Yeah. Good?

So, and this, as we said before, this is just given by some constant and determinant K. This is just some constant determinant K. Yes?

AUDIENCE: Sorry. [INAUDIBLE]

PROFESSOR: OK.

AUDIENCE: Why did you have to go to the Hamiltonian to put that 1 minus i epsilon?

PROFESSOR: Right. It's because that's our previous rule. Because we say, in order to derive this, we use the cheek to take the H, go to-- yeah, H 1 minus epsilon. Yeah. So we want to know how this translate into the behavior in the action. Other questions? Good?

So this is just the same as we discussed before just given by some constant and determinant K. As we said before, that the C is typically divergent. Determinant the K, so typically divergent. But we will see, it doesn't matter. So now we will see, it doesn't-- so, previously, we said, this will not matter. But now we will see it explicitly.

So now let's look at the ZJ. So ZJ is the same integral, x dot K dot x. But, now, with this additional term. So let me just write, again, in the simplified notation, as J dot x. So you view the integration as a huge sum of vector-- yeah, vector product.

So if this is a finite dimensional integral, you say, I know how to do this. We know how to do this because this is just the Gaussian with a linear piece. So we can just write down the answer.

So let's just-- yeah, the rule is that you just treat it as a finite dimensional integral and write the answer for the finite dimensional integral. And then you translate the language in terms of this functional case. So we can write it-- so, again, this would be C divided by delta K. Yeah.

So let me just remind you. Maybe just let me just do a little bit slower. So let me just remind you the standard story for such a Gaussian integral. So if you have dx1, dxn, exponential minus 1/2 xi, Aij, xj plus Ji, xi. So if you have an integral like this, we know how to compute this integral. We can just compute-- include the xi into here by completing the square.

And then, what you get is the following. After you complete the square, you just get the original Gaussian integral. And so, what you get is you get the 2 pi D over 2, or your previous case, delta a. And then, the results you get from the complete integral is Ji A minus 1 Ij, Jj. So when you complete the square, you get the additional term. That's what you get. And this is coming from doing the Gaussian integral.

So now we just have an infinite dimensional version of this integral. And we can just write down the result immediately. We can just write down the result immediately. So we just copy that thing. So we have C. So this C will be the exact the same as that C, because this just comes from doing a Gaussian integral as if J is not there.

So we have the same C. We have the same delta k. Then, according to the rule there, up to the i, which is you have to put in, then we get 1/2 i. Then you have-- then we should have J k minus 1 J. So this is essentially that. You take the inverse of A. So here we get that.

And this, if we translate back into this kind of function language, so this just gives you C delta K exponential i divided by 2. Then you have dt, dt prime. Then you have Jt, K minus 1, t, again, t prime. So K minus 1 should be understood as the inverse of this K and Jt prime.

So the K minus 1 is defined as follows. So the K is-- so you just, again, is the function generalization of the matrix case. So you just have t prime, K, t, t prime, and K minus 1, t prime, t double prime should be equal to delta t minus t double prime. So that's how you define the K, K minus 1.

And this is like a matrix product. Just now you treat the t prime-- yeah, t prime, you sum over that, and then, yeah. So this is just like you have kmn, k minus n, and k equal to delta mk.

You just translate the n into the integral. And the t is corresponding to m. And the t double prime corresponding to K. And the delta function corresponding to that. So that's how we define the K minus 1. And so, this is the result for ZJ.

So, now, the physical object is this object is the DJ divided by Z0. Because you get the expectation value, we always need to divide by Z0. So now, if we take the ratio, so now I can erase this. So if we take the ratio, ZJ divided by Z0, we find all these factor canceled.

So this factor cancels with that factor. So it doesn't matter. So we just get exponential. So, let me, again, using this shorthand notation, i over 2, J K minus 1 J. So this is the physical quantity.

And when we expand this in powers of J, then we get the correlation-- then the coefficient of J give you correlation functions. Or we can just take derivatives. Yes.

AUDIENCE: Yeah, so last time I thought you said the C and the determinant of K can be infinite. So is it OK to divide infinity by infinity and just say it's 1 in this case?

PROFESSOR: No, it's not one, because they are actually-- yeah, as you do in your Pset, say, if you have a free particle, that ratio is actually-- you can calculate to be a finite number.

AUDIENCE: Oh.

PROFESSOR: Even though their ratio is actually, both are infinite. But the ratio is actually a finite number. Yeah. Yeah, same thing with the harmonic oscillator. Yeah.

Yeah, but the key thing is that we actually don't need to worry about them. They just cancel. Yeah. Other questions?

So now, this is our final result for the harmonic oscillator. And except we still have to invert this K. We still have to invert this K. But, in fact-- but, before we do that, first we can see what is this-- whether there's any physical interpretation for this K minus 1.

So let's just consider the following situation. So let's consider a two-point function. So, first, from here, you can immediately see, the one-point function is given by what, the vacuum one-point function of x? So can you see what is the vacuum one-point function for x without doing calculation?

AUDIENCE: 0?

PROFESSOR: Yes, 0. So the reason it's 0, it says because if you get one-point function, you take one derivative with J. So when you take one derivative is J, because of here, it's J square. You will bring down a factor of J. Then, when you set the J equal to 0, and then that will be 0.

So the one-point function automatically is 0. And that's consistent with our expectation. In the harmonic oscillator, the one-point function of x is always 0 because x involve a or a dagger. When you sandwiched between two zeros, it's just 0.

But now, so notice, the non-vanishing one is the two-point function. So now let's consider the two-point function. So two-point function by definition should be the Feynman function because this is a time-ordered product.

So the two-point function, by definition, is the Feynman function. It's the G2. And so, this is given by just this expand-- yeah, just 1 over Z0 i square. You take ZJ, two derivatives. delta Jt, delta Jt prime, and then you take J equal to 0. So the-- yeah. So the two-point function is a Feynman propagator [INAUDIBLE].

So, here, we can just see what we get from here. So when you take two derivatives on this, you take two derivative on J again. So the first derivative on J you bring down a factor of K times J. And your second derivative, we can do two things. You can act on the exponential again. And then you can act-- or you can act on the J factor which you bring down the first time.

But you have to act on the factor of J you bring the first time, because you have any free J left. And when you set J equal to 0, there will be equal to 0. So the both derivative should act on this J, which come together. So, here, then you get minus i, essentially, K minus 1 t and t prime. Just take these two derivatives, then you get K minus 1.

So now, we learned something nice is that this K minus 1, it's actually the Feynman propagator. We discussed before, the Feynman function, we discussed before. Yeah, the harmonic oscillator version of the Feynman function. So, previously, we defined for the field theory. So this is the harmonic oscillator version of the Feynman function.

So we find that K minus 1 t, t prime is just equal to-- actually, i GF t, t prime. So you find that this is just given by GF. So, and then, we learned that the ZJ divided by Z0 is just equal to exponential now minus 1/2 J GF J. So now, it's just GF. It's just everything determined by this GF.

So now we have a consistency check. Now we have a consistency check because, previously, we have discussed that the GF should satisfy certain differential equation. And here we have a differential equation for K. So K minus 1 also satisfy a differential equation. It just satisfy this equation.

And so, now, if you plug in there and into here, OK if by definition, so from this equation. So let me call this equation star, star, star, and this equation star.

So from equation star and star, star, and we find that K minus 1 should satisfy the following equation of the partial t squared minus omega 0 squared plus omega 0 minus i epsilon GF. Yeah. K minus 1 should be equal to the delta t minus t prime. So let me just make sure I get the sign correct. Yeah. So, yeah. Yeah, I think this is right.

And now, if you plug in this expression, and then you find that this equation is actually exactly our definition of the Feynman propagator before. Here we don't have spatial derivatives. But if you look at it, in particular this i epsilon is precisely the i epsilon previously we need to use to define the Feynman propagator.

And now we find that the epsilon prescription, which we previously used as a trick to define the Feynman propagator is actually recovered by this procedure-- recovered by that procedure of H goes to, yeah, 1 minus i epsilon. Just everything is consistent. Just now you precisely recover that procedure. Any questions on this? Yes.

AUDIENCE: So I guess in that expression right there.

PROFESSOR: Yeah.

AUDIENCE: If J is a function, if you just integrate it over t and t prime? Exponential negative 1/2 J dot GF dot J?

PROFESSOR: Oh, you mean this expansion?

AUDIENCE: Yeah.

PROFESSOR: Yeah. So if I write it explicitly, you just have, yeah. Let me write it explicit since this is a very important equation. Yeah, so this is just dt, dt prime, Jt, GF, t, t prime, and Jt prime. Yeah. Good.

So this is all consistent. So this i epsilon prescription, which we did here, automatically recovers the i epsilon prescription in the definition of the Feynman function we defined before. So it's very nice.

And, in particular, so in momentum space, if you find the momentum space, GF omega, if you go to the momentum space, and then you find just equal to i omega squared minus omega 0 squared plus i epsilon. So you see this is actually the previous.

So if you compare with our previous expression for the Feynman function, when you set k square to 0 and replace the omega 0 square by m square, then that's exactly the one we derived before. Good.

So now we can try to find the n-point functions. So now we can work out the all n-point functions. So now you can immediately conclude from the way we do the one-point function that all n-point functions, so general n for general odd n Gn is always 0. Gn is always 0.

So the reason is the following. When we carry out this procedure to take the derivatives, so because the J is always paired in this exponential. So when you take one derivative, you're bringing down another J. So because, in the end, we set J equal to 0. So you have to get rid of all these J which you bring down from the exponential. And then that means, n has to be even.

So if n is odd, then there's always one J left. And when you set J equal to 0, and then will be 0. So this is also consistent with your experience from harmonic oscillator because, in the harmonic oscillator, if you have all the number of x, then you have all the number of a and a dagger.

If you have all the number of a and a dagger together, there's no way when they're sandwiched between 0 and they can annihilate each other, and you will-- yeah. So you will always get 0. And so, here, we get it. Yeah.

So now, for the even n, there's also a simple answer. If I have an n-point function-- so let me just write down Gn, say equal to 0 to t, say x hat t1, tn. So for even n, then you see that, again, because in order for this not to be 0, then all the x has to be paired. Then all the x has to be paired.

So, in this case, we just sum-- so, in this case, the answer is just sum over all possible contractions between x ti's. So by contraction we mean, so if I have xti, xtj, we say there's a contraction between them. And so, that's just defined to be GF xi xj. Oh, no. ti, tj.

So you just pair all of them. So each pair is a contraction. You just sum over all possible contractions, pair of them, and each pairing gives you a Feynman function. So each pairing gives you a Feynman function.

And so, this is actually, in the early days of quantum field theory when you don't have a path integral, to show this is actually non-trivial. Because imagine if you do this time ordered product. There are many, many pieces. Because if you have an n-point function, then you have to write down all possible orderings between them.

But, in the end, it's a very simple result. You just sum of all possible contractions. And each pair is the time ordered. And so, these the early days without path integral is actually a highly non-trivial result.

And so, this was first proved by Wick. So this is called the Wick theorem. But now, we see, it's actually, if you know the path integral, then it's a trivial consequence of that. The path integral is actually quadratic in J. It's quadratic in J.

So, yeah, so give you an example. So let's look at four-point functions. So if you look at four-point functions.

So I don't even have to write that thing down. Let's just draw four dots below to the four points. And then, I just sum over all pairing between them. And each pairing will give me a Feynman propagator.

So 1, I compare 1 and a 2 or 3 and 4. And I can also pair 1 and 3 and 2 and 4. And I can also pair 1 with 4 and the 2 with 3. Oh, 2-- 2 is here. So 2 with 3. I can also do that pairing.

And so, if I write it in terms of the expressions, then I have GF t1, t2-- let me just, t1, t2, GF, t3, t4 plus GF, t1, t3, GF, t2, t4. Then plus the GF, t1, t4 plus times GF, t2, t3.

So you just sum over all such pairings. And each pairing gives you a GF. Yes.

AUDIENCE: Sorry. I thought because of the time of ordering, you can't choose your pairing. There's only one way to pair.

PROFESSOR: What do you mean?

AUDIENCE: Why is it ok to do different pairings even though it's-- things are time ordered?

PROFESSOR: Oh, what do you mean, you cannot do the pairing?

AUDIENCE: The x's are time ordered.

PROFESSOR: Yeah.

AUDIENCE: The x's.

PROFESSOR: Yeah.

AUDIENCE: So wouldn't you have to pair them in that order?

PROFESSOR: No. So that's the key. You just, somehow, this is the consequence of the theorem. If you want to just write down the orderings, then it's actually rather complicated. Yeah.

But, somehow, the magic is that once you, say, write everything explicitly, you do all the everything, in the end you can group everything just in terms of product of the Feynman functions. Yeah. Yes.

AUDIENCE: So for the harmonic oscillator, we know that at a given point in time, the position is Gaussian. And so, that would mean that the n-point function for all the t's equal to each other should be non-zero only for n equal to 1 or 2 but not for n greater, right? Because only the first two moments of a Gaussian are non-zero. That seems inconsistent with the prescription over here.

PROFESSOR: Sorry. Why are you saying that?

AUDIENCE: At a given time, the--

PROFESSOR: No, but all these time are different.

AUDIENCE: Right, but if I were to take the times to be equal.

PROFESSOR: OK, yeah.

AUDIENCE: Then so, for example, the variance of the particle position would be your two-point function evaluated at t and--

PROFESSOR: Yeah.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah, this is just the x to the power n.

AUDIENCE: Right.

PROFESSOR: Yeah, the x to the power n is non-zero for even power.

AUDIENCE: Yes, but then, so then, in this formula, that'd be GF t comma t.

PROFESSOR: Yeah.

AUDIENCE: Which is non-zero.

PROFESSOR: Yeah.

AUDIENCE: But--

PROFESSOR: Yeah, but the GF is non-zero.

AUDIENCE: Right, but then, what I'm saying is, the four-point function then would also give you something non-zero.

PROFESSOR: Yeah, it is non-zero. Yeah, it's very consistent.

AUDIENCE: But the fourth moment of the Gaussian is 0.

PROFESSOR: No. The fourth moment of Gaussian certainly is non-zero. Yeah, x4, yeah, Gaussian you have x4. That's certainly non-zero.

AUDIENCE: But the means is 0, so all cumulants higher than 2 are zero for a Gaussian?

PROFESSOR: No, no, no. For the Gaussian, it is all how equivalent can be expressed in terms of the sigma and in terms of the two points. Yeah, here is just exactly what you show here. Yeah. Just everything can be expressed in terms of the two-point moment. Yeah.

So if you set all the t to be 0, so this is GF is the same. So this is GF. Essentially, the only difference between them. So this is GF. Yeah, essentially you just add them together just, read GF squared. And, yeah. Yeah, just do this exercise yourself, and you will see it's the same. Yeah. Good.

So now, with this preparation in quantum mechanics, now we can immediately just move in field theory. So now we can immediately move to field theory. So now you can say time ordered functions in field theory. So, before we do that, any other questions?

Good. So, again, to go to field theory, now the only thing we need is just copy notations. You just need to change the notations. Just remember how you-- yeah, just replace the appropriate dynamic variable in quantum mechanics by the appropriate dynamic variables in field theory. And then that's it.

So let's write down. So now we have, consider, in field theory, suppose we consider this n-point function. And now, xn-- now xn denote the space time point now.

So let me call the-- previously we call the vacuum for the field theory to be omega in the interacting theory. So now you can see that this quantity, phi x1 and the phi xn, let's look at this endpoint function between the vacuum.

So now, again, if we call this thing to be X, So now we can immediately write down the answer in field theory, just the D phi. Now you just replace the DX by D phi. Then you'll, again, you just have this X, capital X. And then you have now i S phi then divided by D phi without i S phi.

And another boundary condition is that the phi, when you do this integral, again, t goes to infinity to minus infinity to infinity. And now tx will-- should go to 0 for both t plus minus infinity.

So that's the analog of a previous simple x equal to 0. And because it's here, remember, the x here is just the labels. And so, for each variable we require, it's go to 0 at t plus minus infinity. You could do 0.

And also, normally, in field theory, we assume, yeah, in order for the integral to have well-defined behavior, et cetera. We often just assume also go to 0 in the spatial infinity. Yeah, this is just often for convenience.

And physically, this also means that infinitely far away, and we assume that the field are not excited. Yeah, we are interested only in the, yeah, physical excitations in finite region. Anyway, so this is the condition we impose when doing the path integral in the field theory case.

So now, again, you can introduce a generating functional. You can just, again, just copy notation, copy the previous formula. And by changing notation.

So now the generating functional ZJ is defined to be D phi exponential i S phi. And then, now you just add-- now you integrate over all space time points and phi x. Again, we introduce a J. But now you integrate over all space time points.

And then, similarly, the ZJ divided by Z0, so this we call-- yeah, the Z0, again, is just the integral without x, any x. So divided by Z0 is equal to now the omega and time-ordered product of the exponential i.

So, again, just given by that. And, again, this time-ordered product, time-ordered exponential should be understood, as you expanded this in power series, when you expand it in power series, then you have powers of phi. Then you just order those phis in terms of time ordering. And then, again, the integration and J can be pulled outside this expectation value.

Good? And the Z0 is just the same as ZJ equal to 0, so without any J.

So this is, again, just immediately give you a general prescription for calculating n-point function in any theory, in any scalar theory. So, here, I don't even have to specify the precise form of the action. It just carry through.

This also applies to interacting theory. So this formula also applies to interacting theory. And, yeah, it's very general. So this is the power, say, of this path integral formalism. So once you understand in the quantum mechanics case, go to quantum field theory is automatic.

Good. So now let's look at how to treat-- how to do this thing-- calculate this thing in field theory. So, first, let's just look at the free field theory. And then, before we look at the interacting case, let's just look at the free field theory.

So free field theory is almost identical, again, almost identical to the harmonic oscillator case because harmonic oscillator is also-- because the free theory will be also a quadratic Gaussian integral. So everything will be very similar to the harmonic oscillator case. We just need to, again, replace some notations.

So now, let's consider the free field case. Consider this minus 1/2 partial mu phi, partial mu phi, minus 1/2 m square phi square without any nonlinear-- without any cubic or higher power term. And, again, the S can be written-- so S, in this case-- So, here, 0 means the free theory because later we will do interacting theory.

So s, in this case, again, you can write it as-- you can integration by part. You can write it as d4x, d4x prime, then phi x, Kx, x prime, and then phi x prime. And now, the K is given by d squared plus m squared minus epsilon.

Again, this epsilon comes from that thing. You just, if you work through, you just, yeah, it just minus epsilon. And then delta 4x minus x prime.

And, yeah. Good. So, again, we can just-- shorthand notation, so this has a minus 1/2 phi dot K dot phi. Just now, the only difference is that you change it from integration of dt, integration of the full space time. Yeah, everything else is the same.

And. Now, this is ZJ. ZJ just given by D phi exponential, again, just given by a Gaussian integral. So let me see whether I can squeeze in a Gaussian integral, phi dot K dot phi, then plus i J dot phi. So we can, again, write this in the simplified notation as i J dot phi.

So, again, you just-- and then you find this-- you get some other C. Again, you get determinant K. And then you get exponential i over 2 J K1 J. So everything is exactly the same.

And then, this is the same as Z0, this part. And then, again, they cancel. Again, they cancel. So, again, we find-- so, in this case, again, we find that K minus 1 x, x prime equal to i GF x, x prime.

So this i epsilon prescription, so it's precisely, yeah, if you check the definition, so this i epsilon prescription m square goes to m squared minus epsilon, is also the precisely what we did before for the Feynman function.

So, now, the final answer, the ZJ divide by Z0, yeah, I almost don't want to copy it. It's just exactly the same as this. You just replace the dt integral by d4x x integral. And the d4x integral. So, yeah, let me just write it. Exponential minus 1/2, then J GF J. Any questions on this?

And, again, if you calculate n-point function, then you just get the identical structure as here. Just the identical structure here. The only difference is that you replace GF, t1, t2 by x1, x2, x3, x4, et cetera. Everything is just identical. So, to save time, I will not copy them again. So, do you have-- yeah.

AUDIENCE: So in this particular theory, do we have the condition that phi goes to 0 when t goes to infinity?

PROFESSOR: Sorry?

AUDIENCE: In this particular theory, Do. We have the condition that phi goes to 0?

PROFESSOR: Yeah, so that ensures, when you do integration by parts, everything is 0.

AUDIENCE: We have the solution for phi?

PROFESSOR: Sorry?

AUDIENCE: We know the solution for phi in terms of x and t right, and it's like a plane wave?

PROFESSOR: No, this is the boundary condition in your path integral. Yeah, this is the boundary condition in your path integral. We're not talking about this. Yeah, so this is-- yeah, phi just the-- yeah, here, I didn't write down any explicit solution for phi. Sorry. What plane wave we are talking about?

AUDIENCE: Oh, I was thinking about the solution for all phi field theory that we did a few lectures ago.

PROFESSOR: Right.

AUDIENCE: So that [INAUDIBLE]?

PROFESSOR: Yeah, no. No, that will also go to 0. Yeah, no, no, no. That's an operator equation. That's an operator. Here, it's just the field in your path integral.

So, here, you just integrate over all possible. There, when we write that, that's an operator equation with a and a dagger there. So here is just the ordinary function of space time, which we impose the boundary condition in the path integral. Yeah.

And, similarly, in here, there's Wick theorem, just everything just goes through. Everything just goes through. Good. Any other questions?

So, for the last few minutes, then we can venture a little bit into the interacting case. Was there any question? So now we can venture into interactions. So now we have our master formula. And now we can treat what happens in the interaction case. So now we can go to interacting theory.

So in the interacting theory, let's just consider, say, the case which L equal to L0, then you pass some polynomial. For example, the simplest case is just lambda we discussed before to the phi 4. So plus some higher power term.

So, for simplicity, I will just-- but what we will do, we will not depend on details form. Let me just write LI. Imagine you have some interacting terms. So, in this particular case, the LI is equal to that. But we can consider the more general case. Just you have some extra, yeah, something depend on phi.

And, similarly, your Hamiltonian will also be the free theory Hamiltonian plus a interacting one. And the interacting one-- so let's, for simplicity, then this LI, as in this case, only depend on phi, does not depend on the, say, the time derivative of phi. So if it does not depend on the time derivative of phi, and then, essentially, LI, HI is essentially just minus d3x LI.

When you do the Legendre transform to go from, say, L to H, don't change this term if it does not contain time derivative. So, yeah. So, and then, there's a very simple relation between this interacting term in the Lagrangian and also in the Hamiltonian. And this is a free theory Hamiltonian.

And now we will write our total action in terms of the free theory part and the interacting part. And the interacting part is just given by d4 x LI. It's also the same as minus dt HI. Good? So just to set it up.

And now we want to calculate. Again, we want to calculate this n-point function. Which did I erase it? Again, we want to calculate this n-point function. So this is the object we are interested in. And, again, we can just consider generating functional. And we can consider generating functional. Yeah.

Before, actually, we do that, let's just consider-- yeah, let's just consider this n-point function. Yeah, we actually have one minute. We cannot do really much. Let me just tell you the basic idea. And then we will elaborate next time.

So, now, let's imagine we want to compute this object. Again, we just use that formula, this Gn is equal to D phi X exponential i S divided by D phi exponential i S. So now, what I will do is, I will-- now, my S-- so the idea is the following.

Now, this part, integrals are not doable. Because once you have these non-polynomial terms, or non-quadratic terms, we don't know how to do the integral. We don't know how to do this integral even for one-dimensional integral.

So not to mention the path integral. We also don't know how to do such integral for a harmonic oscillator. But, yeah, same thing. We don't know how to do it for field theory.

So, as we said before, even though we don't know how to do this integral, we can treat this perturbatively. So we treat this as a main term. And then we treat the lambda small, then we try to expand the power series of lambda.

And now we can write the path integral as the following. D phi x, then we add S 0, then i SI, and then divided by D phi, exponential i S0 plus i SI. And, now, what we are going to do is, we just expand this term in power series. When we expanded this in power series, and then, essentially, we are reducing it to the path integral of the free theory.

So, essentially, the upstairs and downstairs can be imagined as we are doing the free theory-- now this is a free theory vacuum. We can just view this as an integrand of exponential i0. So the upstairs just become-- so this just become the-- yeah, t x exponential i SI, now in the free theory. And then, downstairs also become 0 t exponential i SI in the free theory.

And now we can just evaluate, in the free theory, such kind of correlation functions. And we evaluate them by expanding the SI in power series. And then just-- so everything becomes just doing some Taylor series expansion in the inside of the integral and then become very simple. You don't need any fancy stuff. First year your calculus you can do. First year calculus you can do.

So when you expand that stuff, still you get something very complicated. And then you can use diagrammatic rules to simplify them. And that's called the Feynman diagrams. And now we can-- yeah, next time we will talk about Feynman diagrams to simplify such kind of expansion in power series.