[SQUEAKING]

[RUSTLING]

[CLICKING]

PROFESSOR: So I hope you all have had a great spring break. So before the break, we talked about Dirac equation. So we introduced the equation. We showed that it is Lorentz covariant.

And then we started talking about the classical solutions of the Dirac equation, because the finding the complete set of solutions for Dirac equation is the necessary step for quantizing it. So let us consider the classical solutions of Dirac equations.

And recall the Dirac equation is given by the following. So this gamma matrices-- so these gamma matrices are 4x4 matrices, and psi is a four complex vector. And we already just from setting up the Dirac equation and the properties of the gamma matrices that this should have solutions proportional to plane wave with k square equal to minus m square.

So our goal is to find the pre-factors. So here, we separate into two kinds of solutions. So this just reminds you-- we discussed that at the end of last lecture-- we call uk corresponding to the positive.

So we always take k to have positive time component. And so this is the positive energy solution. Then we have also so-called negative energy solution. So these are just names.

Does not really means that they have positive energy or negative energy. It just means that the sign here. So now, if you plug this into the Dirac equation, and then you get the algebraic equation for u. You get the algebraic equation for u, and then you find i k slash m u equal to 0, and the i k slash plus m v equal to 0.

So u and v satisfy those equations. And so I suppose you're already familiar with the rotation-- so this k slash. So those equations, in principle, you can just solve them.

So these are just linear equations. And yeah, in principle, you can just solve them. But still there are tricks how we solve them so that it's mostly transparent. So there are two methods of finding the u and k.

So one method is we first go to the rest frame-- say we take k mu, say, to have 0 spatial momentum. And then the k mu will be just equal to m 0. So in this case, those equations become particularly simple.

And also before we solve them, actually we should also specify the gamma matrices we need to use. So the gamma matrices we will use is given by the following. Again, each are the 2 by 2 block.

So this is the gamma matrices we are going to use. And as we discussed before, there are many, many different choices of gamma matrices. And some of them are convenient for certain purpose. And as we will see, this one actually makes the non-relativistic limit more manifest. So it's conveniently reduced to non-relativistic limit.

And so in the rest frame, those equations simplify-- essentially, u just becomes the eigenvalue equations for gamma 0. So you find-- say, this equation for u-- so let's denote to the rest frame by 0, and then-- so the equation for u just becomes this in the rest frame, and the v given by this.

So essentially, this is just corresponding-- essentially, you see the u and v-- u0 and v0 just corresponding to eigenvalues of gamma 0, because this is gamma 0 times u gives you a constant.

It can be-- move that constant to the right-hand side-- it just looks like an eigenvalue equation. And so you can just easily-- you can plug this in and easily solve them, which we already discussed last time. So for example, you find that there are two independent equations.

So this is a 4x4 matrix. So gamma 0 is a 4x4 matrix. And actually have two eigenvalues, say-- so each u and v have two independent solutions. So you find that the-- say, for example, one solution is just given by 1, 0, 0, 0. u2.

So this-- we all just discussed at the end of last lecture here-- I'm just trying to remind you. So in this rest frame, the solutions are particularly simple. And so essentially, you have two independent solutions.

And then you take and take actually arbitrary linear superposition of them because we are solving linear equations-- similarly for v. And you can take the arbitrary linear superposition of these and then you will solve them.

And then for general uk and vk, you can just perform a Lorentz transformation. So you can perform Lorentz transformation. Say, the uk is obtained by S lambda uk 0 u0 and vk is given by S lambda v0.

So the lambda-- so remember, our notation, S lambda, is a Lorentz transformation on the spinors. So the Lorentz transformation on the spinors. And the lambda is just determined by the Lorentz transformation to take this k0 to k.

So for arbitrary-- so we take k0, so we make the Lorentz transformation to k, and then the corresponding uk. Then, it's getting by that. But in practice, if I give you an arbitrary, say-- yeah, give you arbitrary k, and you first work out the lambda, then you have to work out this S lambda.

So working s lambda is possible, but again, takes some effort. But there's still a lot of shortcut. We can actually find the solution. So this is one way to do it, and there's alternative way.

So before I talk about this alternative way, do you have any questions? So this is more or less where we stopped-- so at the end of-- before the break. Any questions? OK, good.

So alternatively, we can actually find a simpler way. So the simpler way is by observing an identity-- you say, if you look at the k slash square-- so you have done something similar in your pset-- if you look at the k slash square-- so this quantity, by definition, is k mu gamma mu k nu gamma nu And then you use the properties of gamma mu and gamma nu, and then you can easily convince yourself this is just giving you the k squared.

This just gives you the k square. And then-- because we are looking at on-shell solution, so this is just equal to minus m squared. Then, this means that if I have ik slash plus m-- which-- so up here in the equation for u and v are plus or minus m.

And then if I multiply by i k slash minus m-- and then essentially I have minus k slash square minus m square, and this is actually equal to 0 because of this equation. So now we can immediately observe-- so now, we can immediately observe if I take a solution-- if I just-- to solve that equation, you can just write u as say some normalization constant-- N will be a normalization constant-- and i k slash plus m.

And then they're acting on some arbitrary u tilde-- some arbitrary spinor. And then this equation-- that equation automatically satisfies. And similarly, v, if I take some normalization constant, and I take minus i k slash plus m-- say some v tilde-- so that equation, again, automatically-- the v equation automatically satisfied because of this identity.

And now, we know that there's only two independent equations-- two independent solutions for both u and v, so we can choose a basis for u. And just by taking this to be, say, us 0-- and then this would be a basis-- a complete basis of solutions. So s would go 1 to 2. So because we have-- and for each of them, we have two independent solutions.

And so the reason we can choose us here-- you can also check that when you take the m-- when you take the to be-- yeah, so when you take k to be k0-- so you can easily check yourself that this equation-- this essentially just reduced to a us 0.

Yeah, this is consistent-- just say when you take that-- you just reduce to this basis. So n is a normalization constant. And so the us vs then provides a basis-- then this is a basis of solutions. And we can be sure these are the complete basis because that's the only-- yeah, essentially, we already know the solution at the rest frame, and we just have the right number of the independent solutions. Yes?

STUDENT: How do we know the normalization constant is the same for uv?

PROFESSOR: Oh, yeah, here, we don't know. But you can guess why it should be the same. It's just essentially-- so this u-- yeah, because the equation looks pretty symmetric and the normalization constant don't depend on the sign of m, essentially. Yeah, in principle, I should write in 1 and 2. Yeah, just-- but in the end, they are the same.

Good. Any other questions? So you can also-- yeah, if you take the conjugate, then you can get the-- what's the u dagger or the u bar or v dagger v bar.

So for example, you just take the conjugate, then you find the us bar. So I hope you still remember the notation-- the us bar-- which is the us dagger times gamma 0 and the us bar-- so this is for the conjugate solution-- will be an-- the same n. We can take n to be real-- say, us bar 0, then i k slash plus m.

So you just use the properties of the-- so you can also find this. So now you can easily check yourself again using this equation-- you can check that the u bar and the v bar are actually-- u and v are orthogonal in the following sense.

And ur bar vs is equal to v bar r us is equal to 0 for arbitrary r, s equal to 1 and 2.

So u and v are orthogonal. And we can also -- and then we need to fix the normalization of u and v. So it's convenient to fix the normalization as follows.

So we will use the following convention-- say us bar k-- say, ur k u bar and us k equal to i 2 m delta rs. And similarly, vr bar k and vs k - i 2m. So these 2m just for convenience. So it just convention for convenience.

And so you can check yourself that actually-- yeah, so different u-- different value of r and s, they're also orthogonal to each other. You can already see already in the rest frame. You can just easily check using those expressions, but you can also check again using this identity that the different value of r and s, they are orthogonal to each other.

And when they are the same, and then we normalize the constant to be 2 m. And from here, you can deduce that N is equal to 1 over square root E plus M. So from now on, I will also use this notation for E.

So the E-- the energy is omega k. So now, I will also use this notation. Good. So this gives you a complete basis of solutions to the Dirac equations.

So now, let me just write them down to give you a little bit intuition about the explicit expression. So with those gamma matrices, we can just plug in here. And we know what the us 0 vs 0. We know n, so in principle, we can write them down explicitly.

So you find that in that basis, then the us k is given by the following-- given by the square root E plus M xi s, and the sigma dot k divided by square root E plus M xi s.

And the vs k is given by sigma dot k divided by square root E plus M also xi s, and E plus M xi s. And xi s is a 2. So s from 1 to 2, and xi 1-- so psi s are two component vectors. One is given by 1, 0, and the other is given by 0, 1.

So that you get-- so you can see in the rest frame, k equal to 0-- essentially, for u, you're just left with the upper half because this is 0. And then we have the 1, 0 and the 0, 1. And then when k equal to 0-- and for v, just left with a lower half.

And again, we have the 1, 0 and the 0, 1. So this is consistent with the rest frame result we obtained earlier. Good. Any questions on this? Yes?

STUDENT: What are the sig -- what's sigma?

PROFESSOR: Oh, sigma just the sigma matrix. So this sigma just Pauli matrices. Sigma 1, sigma 2, sigma 3-- Pauli matrices. Good?

So let me just mention a few-- make a few remarks on the solutions. And to be familiar with the properties of the solutions actually will be very useful when later you do Feynman diagram calculations, because the different-- when you involve, say, electron fermions-- and now, it's no longer plane wave.

So this u and v will enter your Feynman diagram calculations. And if you are familiar with their properties-- and will help you to do such calculations. So let me just first mention some properties. Make some remarks.

So first is that if we consider the non-relativistic limit-- the non-relativistic limit-- so that's corresponding to-- you take the magnitude of k divided by m goes to 0. And then in this limit, E would be approximately equal to m. And then you find the leading order when you expand, say, in-- yeah, so you can expand this in 1 over M. So the leading order, you find that the us k-- so you can just easily do these two.

And yeah, so this term, certainly, you can just ignore because E-- so this is M-- so this is k divided by square root of M, so this is higher order than this one. And here, it's the square root of M.

So to leading order, you have square root 2m xi s 0. Similarly, vs k equal to square root of 2m leading order 0 xi s. So essentially, what you see in the non-relativistic limit-- for example, if we look at this positive energy solutions-- so this just reduced to arbitrary-- so us k-- yeah, now you can do linear superposition-- just uk just reduce to arbitrary two component complex vector.

Yeah, so you just-- so this is just some constant. And as a basis, you just have this basis, and then you can just now do arbitrary-- yeah, just become the arbitrary complex vector. So the should remind you how we describe a spin-half particle in non-relativistic limit.

In the non-relativistic limit, the way we describe the spin-half particle-- and it's precisely a two-component vector. So so this is exactly as the description of a spin-half particle.

Yeah, the wave function-- yeah, here, you should understand this-- the wave function description of the spin-half particle in non-relativistic quantum mechanics. And as I will not go through here, but you can show in the non-relativistic limit, this solution actually decouples. In the end, when you solve the equation, only one half of the equation-- one half of the solution remains.

And actually, this is no longer-- it just gets the positive energy solution. So in that limit, you essentially only reduce to two components. So this is the first remark.

So this is the first indication we see that this should likely describe a spin-half particle. The Dirac theory likely describes a spin-half particle. So the second limit we can consider is considered the ultra relativistic limit.

So this is a limit which E is much greater than-- energy is much greater than M. And E then will be approximate, say, to k-- to the magnitude of k. And now let's also define the direction of k by just k divided by its magnitude.

So we also introduced the k hat is the unit vector along the direction of the k. So in this case, you can do a small m expansion-- just do a small expansion-- this expression for u and v. And then you find the leading order-- us k given by square root of E.

And then you have xi s, and then you have sigma k hat xi s. This is for the us. And for the vs, you find-- so that's the-- so you find that solution simplify, and so that's the result you get.

So it's not manifest from this expression, but actually, there's something very special in this simplified expression in the ultra relativistic limit. So if you look at-- so the expression for u and v-- these are linearly independent, in the sense that no matter how you do linear superposition of u, you will not be able to get v.

And also how you do linear superposition of v, you won't get u. But in this limit, even though it's not clear-- not manifest in this form, but in fact, these two are linearly related to each other. So this can be seen as follows. So if you use the identity that's sigma dot k square-- so this is the analog of this equation for the sigma matrices-- for the Pauli matrices.

And this, you can easily-- again, using the property of Pauli matrices, you can convince yourself that this is just giving you k hat squared. And this is actually-- because this is unit vector, this is just 1. So now, if you use this property, this is equal to 1.

And then we find-- and then let eta s equal to sigma k-- sigma dot k hat xi s. And then you find that the vs k can be written in the exact form as us k. You just replace eta s by-- xi s by eta s.

And then you can have it in the following form-- square root E eta s sigma dot k eta s. So obviously, this is equivalent to that once you plug in the eta s equal to sigma k psi s, because this just gives you that. And when you plug in to here, you get the sigma k square, and you get 1, and again, you reduce to that.

But now, this have exactly the same form as this one, except now eta s is a linear superposition of the xi s. Then that means that the vs is a linear superposition of the us. So this means that the vs is a linear superposition of vs.

So vs now is a linear superposition of vs-- of us. So they are no longer independent, so they are not linear independent. So actually, if you remember one of the ways we have discussed before, this may not appear a surprise to you, because previously, we already discussed when we derive the Dirac equation. So Dirac equation-- in order to satisfy all the properties for non-zero M, we need four dimensional matrices. So that means psi is a four vector.

But we also mentioned that if you in the massless case set m equal to 0-- in the massless case, actually, you can use two components-- just use gamma matrices. So that means that if you solve a massless Dirac equation, you only need-- you only have two independent solutions. And this ultrarelativistic limit is precisely like the massless limit because the mass-- the energy is much greater than the mass, and then you can neglect the effect of mass. So this is consistent-- with that the massless Dirac equation consistent with that-- only requires two components rather than four components. Yes?

STUDENT: So in the ultrarelativistic limit, theyre linearly related, but that relationship v bar times u equals 0 still holds, right?

PROFESSOR: Yeah, that you have to be a little bit careful.

STUDENT: Does that mean that's not to be thought of as an inner product on the spinor space?

PROFESSOR: Yeah, so that's a very good question. So you have to be careful a little bit-- and when you take the limit in deriving that relation. Yeah, just you have to be a little more careful in understanding.

Yeah, I forgot the exact-- what's exactly happened to that equation, but I remember there's some subtleties. Yeah, you have to be. Good. Other questions?

So also there are some other useful relations which will be useful later. So one of them-- so let me also list some useful relations. So why is that if you look at ur dagger and us-- so this is k us k-- again, this is orthogonal between r and s.

And then it's given by 2e dot rs. So here it's bar. So the difference here is bar. And so when you have a bar, you have additional gamma 0.

And then here is 2m, and then here it becomes 2E. And here, you have additional i because you have a gamma 0 because gamma 0-- remember, gamma 0 square 2 minus 1. So gamma 0 has actually eigenvalue to be pure imaginary, so that's why there's a difference in i here.

And similarly, the vr dagger k and vs k equal to 2E delta is. So those expressions to-- yeah, will be very useful later when you actually do calculations. Yeah, I forgot to mention-- yeah, the reason is that-- the reason that limit is subtle when you do that is because you remember the normalization we do here is m, and so you have to change your normalization when you go to the massless case.

Yeah, so there's various things you have to be careful. So you have that, and then you also have the relation that the ur dagger k-- so because everything is on-shell so in principle, I can only-- I only need to specify the spatial momentum. So for convenience, let me just specifier the spatial momentum, which is easier in notation.

So now it turns out ur and the vs-- so now it's a dagger. When you have bar, they're orthogonal. We have the expression there. But then you have dagger, it turns out it's orthogonal to minus k. You have to reverse your spatial momentum, and this is equal to vr k and us minus k.

So this is equal to 0, but this is actually not equal to 0. ur dagger k us vs k is actually not equal to 0. So they're not orthogonal. Only the bar relation is orthogonal, and then this relation is orthogonal.

And similarly with the conjugate of this. So those relations you can just check them using explicit form of those expressions. And we have this correction here. You can just check those expression is true.

And so I will not go into that. But actually, more remarkably, those expressions-- so when we derive this expression, we actually use the explicit form of the gamma matrices. It depends on the explicit form.

So those expressions depend on explicit form of these gamma matrices.

So if you have a different set of gamma matrices, then you get different forms. But of course, physically, they're all equivalent. But it turns out that those relations-- so those relations actually are independent of the choice of gamma matrices.

So let me just label them by-- let me call it star, and here by star star. So star and star are actually independent of choice of representations of gamma matrices. So they're actually universal for any choice of matrices.

So this is very useful because when you do calculations, then you don't have to worry about which choice of gamma matrices you make, and those expressions are true. So you can actually derive those expressions without using gamma matrices-- just abstractly only using the properties of the Dirac equation itself and the general properties of gamma matrices without using explicit representations. Yes?

STUDENT: So is the other normalization one dependent on representation?

PROFESSOR: Which one?

STUDENT: The one with ur bar v.

PROFESSOR: No, that also does not depend on representation. Yeah, that also does not-- or this normalization does not depend on the representation. But this expression somehow is-- yeah, that's just our definition.

We just define them this way. We define them in this way. So this just follows from those general expressions just from this equation. But this equation is less obvious because without gamma 0, then it's less directly related to the Dirac equation itself.

But it turns out this expression also independent of the choice of gamma matrices. Any other questions? I can give you a derivation of this without using gamma matrices. but that take a few minutes, So I will skip that.

You can read it in Peskin. So Peskin has a derivation of this. Good? But you should remember this yourself-- just remember it.

So the first remark is that often we actually need to use the projection-- so you can also construct projections-- projectors to the positive and the negative solution space-- spaces. So consider-- so the projector to the space spanned by us can be defined as the follows. So let me call it p plus. So this is just from 2i m, so that's coming from the normalization.

Then, just sum s equal to 1 and 2 us k and us bar k. So you can easily check yourself because of orthogonal relations-- or because u and v are orthogonal. When you act this on the general solution, we'll just project into solutions proportional that only involve linear superpositions of u.

So now, you can check yourself explicitly with this. So just use various-- use property of this u. You can check yourself explicitly with your-- I will not do there.

And you can check that this is indeed a projector. If you look at the p squared, it gives back to itself. And with a little bit more effort, you can further show that the p plus can be written in the following form. ik slash plus m divided by 2m.

So this requires a little bit more effort so I will leave as an exercise for yourself, or you can get it from the book. And similarly, you can define the projector to the space of v. So the p minus-- you can also define p minus to be minus 2 im because of the minus sign here.

So 1 over minus 2 im some s vs k vs bar k. And you can show that this also have a-- can be written as something like this-- minus ik slash plus m divided by 2m. So again, this I will leave as an exercise to yourself.

So this is the projector to the space of v's. And then from these two expression, p plus p minus-- you easily see that p plus plus p minus equal to 1. And you see the expression I just erased-- so from the-- taking the product and the p plus p minus is equal to p minus p plus equal to 0. So indeed, they are just projectors-- orthogonal projectors into different space. So this concludes our discussion of the classical solutions of Dirac equations and their properties. So do you have other questions? Yes?

STUDENT: So we've seen hints that this corresponds to spin 1/2, but wouldn't it make more sense to just define the angular momentum operator and see whether it has eigenvalues or not?

PROFESSOR: Yeah, that you would need to quantize the theory, but we haven't quantized the theory. Yeah, that would be our next step. So once we have found the full set of classical solutions and now we can quantize it. And once quantize it, then we can actually really find this angular momentum.

Other questions? And indeed, actually, I think that would be one of your pset problems to show yourself that this is spin-half. Yeah, but yeah. Other questions? OK, good.

So now we conclude the discussion. Now we can quantize the theory. Now we can go to the quantum system. So the-- remember, the action for the Dirac is d4x psi bar gamma mu partial mu psi partial mu minus m psi. So this is the action which gives us the Dirac equation.

So in order to quantize this, we first-- remember our procedure to quantize a theory. So we need to first find the canonical momentum, and we also need to impose the canonical commutation relation, and then we expand the operators in terms of complete sets of classical solutions, and then that's corresponding to the solutions to the operator equations. So let's try to find out what's the canonical momentum for psi.

So as in the complex scalar case, we can treat the psi and psi dagger as independent variables because of the-- yeah. So then the canonical momentum for psi-- if you look at this expression. So you look at the partial 0 psi. Then you find-- so the Lagrangian density-- if you look at that, if I write it bit-- I have the form psi dagger.

So this psi dagger gamma 0, and then gamma 0 psi 0 psi, then plus the rest of the terms. So in order to find the canonical momentum, we are only interested in the part of the Lagrangian density which are related to the derivative of psi-- or the time derivative. And so that's the only term involving time derivative of either psi or psi bar or psi dagger.

So this gamma 0 squared equal to minus 1-- so this is just i psi dagger partial 0 psi. So now, we can immediately find out that the conjugate momentum for psi is given by i psi dagger, because you take a derivative on psi 0-- partial 0 psi, you just get i psi dagger. But the conjugate momentum for psi dagger is actually equal to what? So what is this?

STUDENT: 0.

PROFESSOR: Thi is equal to 0 because there is no time derivative on psi dagger. So this tells you-- so this-- so the fact that this-- equal to 0 tells you actually the Dirac system-- the action-- this Dirac system is actually a constrained system.

So this is considered to be a constraint, because you can no longer normally-- so remember, when we say from the Lagrangian, you find the canonical momentum, and then you invert the canonical momentum to find the Hamiltonian. But when you have this equal to 0, you cannot invert it. You cannot express your, say, psi or psi dagger in terms of pi or pi dagger.

And so this is a constrained. So that means you actually have a constrained system. It means you have a constrained system. And frankly, constrained systems are very annoying. So they have involved lots of-- you have a lot of formalism in order to, say, treat constrained system-- to quantize them, et cetera.

But fortunately, in this case, there's actually a simple trick to go around it. And so now, I will not use this full-fledged constrained quantization to do the job. I will just use a simple trick which will reach the same answer. Yes?

STUDENT: Is it like the canonical momenta not well-defined, because I could integrate by parts to move around the time derivative because I can move the time derivative from psi to psi dagger, right?

PROFESSOR: Yeah, so there's ambiguity. So that's part of the reflection that this is a constrained system. So this is actually typical to have these kind of constraints. It's typical whenever you have a first-order system.

So here, it's different from the Klein-Gordon action is because here, we only involving the first derivative-- one derivative rather than two derivatives. And so this is not our standard system. So the way to go around this is the following.

The way to go around it is the following. It's now-- so this equation-- if we look at this equation-- so this equation tells you that the canonical momentum conjugate to psi is actually psi dagger. So also here it's a little bit funny, because you remember in the standard story, the canonical momentum is related to the time derivative. And here, there's no time derivative here-- just psi dagger itself. And this is another indication that this is a constraint rather than-- yeah.

So anyway, here, it tells you that the dagger is, in fact, the canonical moment of psi. And now, we can just-- now in this equation, we can just try to interpret the psi dagger as the momentum rather than as the configuration variable. So now, let's rewrite the Lagrangian.

So the Lagrangian has the following form-- minus i psi bar gamma mu partial mu minus m psi. So this is the Lagrangian density. So we'll rewrite it as the following. So let's first separate the time derivative.

So this is psi dagger. So just this term involving the time derivative. And then the rest, I have psi bar gamma i partial i minus m psi. But here, they are no longer involving the time derivative.

And now, we just interpret this as the canonical momentum for psi. And then this is just something involving spatial derivative of psi. And I just interpret this as the Hamiltonian for psi.

So this expression just involving psi dagger and the psi-- involving some spatial derivative. And so this is essentially just some functions of psi-- pi psi and psi and maybe with some spatial derivatives. So now, you see that this Lagrangian density-- actually, if I interpret the diagram as the canonical momentum, then it has the form of this canonical transformation to go from the Lagrangian density to Hamiltonian. And then I can just interpret this as the Hamiltonian.

So now, I can just interpret this as my Hamiltonian. So we just treat-- so now, we just-- so this is-- so psi dagger and psi. And now, this is the full phase space.

So this is the momentum and this is the coordinate, so this is the full space, rather than previously, we would interpret them as a configuration space. And then the Hamiltonian just equal to i psi bar gamma i partial i minus m psi. So this is the Hamiltonian density.

So we now then have-- now, this just goes back to the standard formalism. Any questions on this? Yes?

STUDENT: So can we still interpret psi dagger as a different field than psi?

PROFESSOR: it's canonical momentum.

STUDENT: So it's no longer two different fields?

PROFESSOR: Yeah, it's no longer two different fields. For example, if there are two different fields, then psi and psi dagger, they should commute. But now, since they become canonical momentum, they no longer commute. yes?

STUDENT: If you take psi bar and then you bring the, you pull out the gamma 0, isn't this just like the right-hand side of the-- like our earlier version of the Dirac equation-- the one where you like [INAUDIBLE]? Like, the one with the alpha and beta, like the other Dirac equation we wrote down earlier?

PROFESSOR: Yeah, that's right. Yeah, it's very-- essentially just that Hamiltonian. Yeah, that's right. Yes?

STUDENT: Is there not a problem with this treating the time differently? Is it no longer Lorentz invariant?

PROFESSOR: Yeah, so when you-- yeah, when we do the client encoding equation in the scalar theory, when you quantize, you always have to treat time differently. And then the Lorentz symmetry come out in the property of the state. Yeah, but you always-- when you quantize, you always have to treat time separately. Yes?

STUDENT: So now we have the momentum operator, its just the complex conjugate of phi?

PROFESSOR: Yeah, exactly.

STUDENT: So this like, its still like constrained.

PROFESSOR: Yeah, it's a-- yeah, this is just-- yeah, so that's why we call this a constrained system. But now, I'm just giving you a shortcut so that we have to don't have to go through this-- yeah, the intricacy of constrained quantization.

So now, the canonical quantization becomes easy. So now, this is my momentum. So now, we just have the canonical quantization. I think today, we may have unfortunate timing. So now, become psi-- is coordinate.

And now, we have psi dagger. We have pi psi, which is the momentum. So this, by definition, should give you the delta function, so equal time correlation function.

So here, we should-- because remember, this is a four vector, so this also have four components. So let me just now suppress this subscript psi because there's only one variable here now.

So here, there's also coefficient-- so this is also four vector beta, and then here should be delta alpha beta. So alpha beta are spinor indices. So if you impose-- and this is equivalent just to psi alpha t, x psi dagger beta t, x prime equal to i delta alpha beta x x minus prime.

And so we can now expand psi x in terms of complete set of solutions. So this is the first step of the quantization. So first is-- this is the commutation relation.

So the second step-- remember, we can expand the psi in terms of complete basis of classical solutions. So psi x just equal to-- again, we integrate over all spatial momentum. Again, we use this factor for convenience as in the scalar case.

And now, we just need to sum over all possible solutions. So here, we have a sum over s equal to 1 and 2. So we have-- then we have ak s us exponential ik x. And then plus bk-- sorry. k should be here-- sorry.

bk s and vs exponential minus ik x. So these are the complete-- because complete set of solution are u and v. And we call the coefficients of a and b, so let me call it b dagger as we did for the complex scalar case before.

So we-- so the coefficient, we call them a and b. Actually, it doesn't matter. We call it b. Actually, let me just call it b. Yeah, so you can just-- you have u and v, and then the coefficient will be a and b.

And so now, you can just plug this. So if this equation is 1 and this equation is 2, and you take the complex conjugate, psi dagger, you can plug it in here to find what's the commutation relation between a and b.

So you find that the-- yeah, so plug 2 into 1. So you find the commutation relation ak s ak prime p, then equal to bk s and bk prime t. Then, equal to 2 pi cubed delta st, delta 3 k minus k prime.

So now-- and you can also define the vacuum. And all other commutators are 0-- with all others 0. And you can also define the vacuum, as we did for the scalar case, to be annihilated by ak and bk. And now, you can, in principle, build your Hilbert space. Yes?

STUDENT: For equation one up there, are you supposed to not have factor pi just because momentum is i psi dagger?

PROFESSOR: Yeah.

STUDENT: And then--

PROFESSOR: Oh, right. Sorry, there's no-- yeah, thank you. There's no i here. Yeah, because the momentum is i psi dagger.

Yeah, there's no i anymore-- just the delta alpha beta. So all look pretty straightforward, but you actually have problems. You actually have problems.

So the first problem some of you may be already asking in your head, is that, are we supposed to describe spin-half particles? But if we are suppose to describe spin-half particles, then why do we find the bosons? So these are commutation relations for bosons. It's because they commute.

So different k-- sorry, I think I'm missing dagger. Sorry. So different ak, they commute with each other. As we discussed in the scalar case, when the a mu all commute with each other, this is boson. It means the particle you created, you can exchange them and don't-- yeah, you can arbitrarily exchange them.

It's symmetric. But fermions, they are supposed to be Pauli principle. So you're supposed to get the minus sign when you exchange them. So this cannot-- this doesn't look to be right.

So this is-- the first thing it seems to be we were finding bosons rather than fermions. And the second is that now if you find that the Hamiltonian-- so now, if you try to find the Hamiltonian-- so H is just the integration of this Hamiltonian density.

And you can just plug it in-- plug that expression for Hamiltonian density in. Where is my Hamiltonian density? Yeah, plug that in, and then you will find the-- yeah so you find just this just given by i d3 x psi bar gamma i partial i minus m psi.

So now, if you plug them in, you just plug them in-- plug this mode expansion in-- and then you find actually-- then you find the following expression-- omega k. And you sum over s from 1 to 2.

And then you find it's given by ak s dagger ak s minus bk s dagger bk s, and then up to some infinite constants. So now, you see something problematic here. So what's problematic you see? Yes?

STUDENT: The energy can be arbitrarily small.

PROFESSOR: Yeah, energy is not positive definite because of this minus sign. So this minus sign tells you that if you excite a lot of these b particles, you can make the energy as negative as you want, and then the theory don't make sense.

And then this is a theory with energy unbounded from below. It's a theory with energy unbounded from below. So that means this vacuum is unstable. So there must be something wrong with this procedure.

So it turns out that there's a very quick fix to this problem, and this quick fix was invented by Jordan. So he recognized that if you don't do the standard commutation relation-- if you do instead of commutator-- here, you change it to anti-commutator.

Remember, this bracket-- this curly bracket means anticommutator. And that means that here, you change it to anticommutator, and then still with the same thing-- and now this becomes anti-commutator. So let me see.

I just want to make sure I get the-- yeah, if we change anti-commutator also from the convention, you change this to dagger. Now you change this to dagger as we did for the complex scalar case. And then you just get this.

And now, if you do that, and then this becomes plus sign. And now, not only this becomes plus sign-- and also all different ak for different s and t-- they anti-commute with each other. They anticommute each other.

So anticommute each other means that if you create a particle-- for example, ak dagger-- and you create another particle, ak prime dagger on 0, and now these two anticommute.

It means when you exchange all, you get the minus sign. That precisely gives you the minus sign in the Pauli principle-- in the statistics for fermions. And so changing this thing to the anti-commutator-- and so solves all your problem.

And your energy now is bounded from below, and then you actually now actually describe some particles which obey the Pauli principle. So yeah, I think we still have 1 minute left, but we-- but it's a good time to stop, because otherwise, other things will take more time.

So let me just make a couple remark why the commutator-- why here, you need to do something unconventional. So if you look at this Hamiltonian density-- so even without doing this calculation, you can already conclude from here that this H cannot be bounded from below for the following reason.

So from the Dirac equation-- let me just erase here. So from the Dirac equation, you have gamma mu partial mu minus m psi equal to 0. So that means that the gamma i partial i minus m psi is equal to minus gamma 0 partial 0 psi. So now, if you plug this equation into here, and then what you get-- you should find that H is equal to i d3 x psi dagger partial 0 psi.

So because the psi-- now this becomes a operate equation, and psi satisfies this equation. And now, you find H becomes this form, which is psi dagger times the time derivative of psi. And time derivative of psi-- remember, psi expanded both in terms of the positive energy solutions and the negative energy solutions.

So depends on which omega you have. You can have either positive omega negative omega, because this is the first derivative-- a first time derivative of psi. And so this cannot be a positive definite. So this quantity cannot be positive definite.

Anyway-- so you have to do something more radical. If these are conventional functions, and then this won't be positive definite. But now, what we did is that we make these into functions which anticommute. When they anticommute, turns out this to be a positive definite quantity.

As the ordinary function, this cannot be positive definite. But if they anticommute, then become a positive definite. So let's stop here.