[SQUEAKING]

[RUSTLING]

[CLICKING]

HONG LIU: So at the end of last lecture, so we discussed this LSZ theorem, which tells you how to obtain scattering amplitude from correlation functions, from time-ordered correlation functions. OK, so if you want to compute, say, some scattering amplitude from alpha to beta-- so alpha's some initial state and beta's some final state. Say alpha consists of momentum p1 and PN-- or pm, and beta, say momentum p m plus 1 and pn.

And then you can get this scattering amplitude just by taking your momentum-space correlation function, OK, for the n points. So all together, you have an n points for the external momentum, OK, and then you take the on-shell limit. You take the on-shell limit, and then that gives you the product of the external propagators, then times the scattering amplitude. OK.

So this is the relation, OK, so here I have stripped out the momentum conservation on both sides. Of course, momentum has to be conserved. Inside, the total momentum has to be zero. And so this limit is the on-shell limit, and in the on-shell limit, the initial momentum-- OK, actually, I should call it minus p. Sorry, it's minus pm.

So the initial momentum, for those with the initial momentum, you take p1 to be-- p1 0 to have minus omega p1, OK? For the final momentum-- so p alpha. For the final momentum, you take the p beta 0 goes to omega p beta, the final, OK, with the plus sign.

OK, so that's how you distinguish the initial state from the final state, because when you obtain a correlation function, you don't distinguish what is the initial and the final state. You just have some momentum. This is a function of some arbitrary momentum.

But the scattering amplitude, of course, those momentum are on shell, and so the way you distinguish the initial momentum and the final momentum is by taking the initial momentum, say to go to the negative root, and the final momentum to take the positive root. But since the alpha, it consists of minus p1, and then the initial state, you have positive energy, OK, you have positive energy.

So this tells you that when we compute the scattering amplitude-- when we compute the scattering amplitude, we should take the-- we just take all the Feynman diagrams which you used to calculate these correlation functions, OK, and then you sum over-- you sum over, say, the truncated-- so you see the relation between these correlation functions and the scattering amplitudes.

So they differ by this product. They differ by this product of the external propagators, OK, so for each external momentum-- so here there's a propagator, and just as if that-- when you get-- so this scattering amplitude was corresponding to this one with all external propagators stripped. OK, so that's why you consider the truncated diagram not including the external propagators. And you take it on shell.

OK, and also, since we are interested only in the process-- which, all particles participate in the scattering process-- so we also only consider the connected diagrams. OK, consider the connected diagrams. OK, so this provides a simplification, so you get fewer diagrams and simpler expression than you would have got from calculating these correlation functions, OK, correlation functions. So any questions on this? Yes?

AUDIENCE: So can you explain again why this diagram, like, you have one branch and then there's a loop?

HONG LIU: Oh, yeah, yeah. Yeah, I will explain that, but before that do you have other questions? OK. So now I will explain a few things.

OK, the first thing is this sign convention, OK, this sign convention. OK, so remember this Gn, so let's go back to the definition of this Gn, these momentum-space correlation functions. So this is obtained by doing a Fourier transform.

Say-- I think it would be minus sign. OK, by doing the Fourier transform-- yeah, by doing a Fourier transform of your coordinate-space correlation function, the coordinate-space correlation function can be written as the following, OK, so you have phi x1 and phi xn. OK.

So now, for those to go to the initial state, OK, for those to go to initial state, we start from alpha to beta, so then you want those phi corresponding to the initial state to act on the right, OK, to act on the right. And then you do the Fourier transform, OK, and then you do the Fourier transform.

So now let's just consider one of them. Let's just say consider you have phi x for the initial state acting on the right, and then you do a Fourier transform. OK. OK, you do a Fourier transform.

And so this, if you just record the mode expansion for phi and the phi contains a and a dagger-- and the a pieces acting on the zero will just give you zero, so only a dagger piece will survive. OK, and the a dagger piece is multiplied by-- say you will have, say, some k, and then have i omega k t plus minus. Yeah, essentially-- yeah, let me just write it in a simple way.

You just have exponential i k x. OK, I think it's the exponential minus i k x, so you just get the exponential minus i k x. OK, with k it's the on-shell momentum, so k is given by omega k and k. OK. So when phi x acting on the zero, you keep the part which is corresponding to the a dagger, and then you get a piece proportional like this.

OK, and now when you do the Fourier transform and then you find just p, then your p just equals to minus k. OK, so that's related to the minus sign there, OK, related to the minus sign there, and also, when your p equal to minus k, then that means p0 is equal to minus k. OK, so that's where that sign in the initial state come from.

And the same thing with the final state, so for the final state, then you need to look at the phi x acting on the left to this, and then you do the Fourier transform. OK. You do the Fourier transform, and then in this case, this becomes-- on the left, it's the A part acting on the left, so this gives you k exponential i k x, OK, is the a acting on the left.

And then when you do the Fourier transform, then here it gives p equal to k. OK, so that's why in the final state, you just have p equal to k, and then p0 is equal to just omega k. OK, so this explains the sign.

So this explains the sign. It's just from whether you act on the initial state or act on the final state. OK. Good? So-- yes?

AUDIENCE: Right, so time-ordering of x1 to xn right here?

HONG LIU: Yeah. It-- of course, when you derive that, it matters, but here, for this argument, it doesn't matter. It just has to act to the right. Yeah, for the initial state, you have to act to the right. Yeah, of course, to derive that theorem, the time-order matters. OK. Yes?

AUDIENCE: So this was just for free scalar field theory, that you showed that this is true?

HONG LIU: No, this is not for the free scalar field theory. Yeah, it's the-- here I just detail you the sign convention. OK, you can do this for the full interacting theory.

Yeah, just use the free scalar as the-- yeah, because when you go to plus or minus infinity, you can just reduce to the free particles. OK. Other questions? OK, good.

So this is the first comment. The second comment, each side, we need to here-- from here, we need to truncate the external propagator. So we mentioned that if you have a diagram like this, OK, if you have a diagram like this, say minus P1, minus P2, to P3, P4, for a diagram like this, OK, and then the square root and amplitude are just given by minus i lambda, because you need to throw away all the external propagators. You don't have to worry about the external propagators. OK, you just need to-- you have to truncate all the external propagators, so that means you also throw away diagrams like this.

You can consider arbitrary, complicated diagram, OK, as far as that only happens-- yeah. Yeah, yeah, here, it should-- I don't draw very well, so this just touch at the one point, OK? So all these diagrams, you can ignore them, OK, because they're just corresponding to the correction to the external propagator, OK, because they don't touch with the-- they only-- yeah, you can also-- yeah, you can do it on any of them, OK? So such kind of diagrams, they only change the external propagator, but since we truncate the external propagator, and then they don't matter at all, OK, and it's all included in this diagram. OK.

So the reason is the following. The reason is the following. All this diagram do is just modify the properties of the external propagator, OK, modify the properties of the external propagator.

And the only way all these diagrams can modify the external propagators is to give you an overall constant, OK, so that's what this Z is corresponding to. So this Z essentially just captures all these different corrections, OK, all these different corrections. And now you have truncated them, and so you don't need to worry about them.

OK. So Z also has an expansion, just 1 plus order lambda, et cetera, so the leading order, so Z does not contribute. OK, you can just set it to 1, and when you go to the higher order, then the Z can make a contribution. OK, so you just need to separately take into account the Z, and there's no need to contribute-- to calculate those things separately. Yes?

AUDIENCE: We also truncate diagrams with loops in two different legs?

HONG LIU: Yeah. Yeah, yeah, any of them, you can do any number of-- as far as they only concern external legs, it's fine, yeah, because this only-- any of those corrections only concerns one leg, yeah. Yes?

AUDIENCE: But when calculating the actual, let's say, n-point function, we have-- we're supposed to include all of this?

HONG LIU: That's right, yeah. When you calculate the n-point function, you have to include all this, but when you calculate scattering amplitude, you don't need to.

AUDIENCE: Don't need to, all right.

HONG LIU: Because in the scattering amplitude, you literally divide by the external propagator, and all of these things, they-- yeah, all these things, essentially, they just modify the-- give you the correction to the external propagator and include it in that constant Z, yeah.

AUDIENCE: So is Z the same or different for different processes?

HONG LIU: Yeah, Z is the same. No, Z is the same for different processes, but it's different for different particles. So here we only have one type of particles, so we only have one Z. But if you have two kinds of particles, then z is different for different particles.

AUDIENCE: Oh, OK. So for example, for, let's say, a fixed process, let's say, like, 3 going to 3, Z would be constant for all choices of momentum, right? Am I correct to say that?

HONG LIU: Yeah, yeah, yeah, it's all-- yeah, a constant for all of them, yeah.

AUDIENCE: OK, yeah, because if it depended on momentum, then it would be kind of useless, right?

HONG LIU: No, no, no, it does not depend on momentum. Yeah, this just-- correct. You see, all such things don't change the momentum. OK, the momentum don't change. Yes. Yeah. So we will not go into details of the Z. And that is in the QFT2, and so we will discuss how to calculate this z in QFT2. But the leading order, they don't matter, and so we will start with 1. So for our purpose, actually, it's not important. Yes?

AUDIENCE: Is there a physical interpretation for what interactions are included in Z?

HONG LIU: Yeah. Yeah, this is the self-interact-- yeah, just when the-- when you have an interacting theory, so the particle can interact with itself, when the particle propagates, it actually can interact with the virtual particle. It just all comes from this kind of diagram. You can have single particle, and you can have such a diagram like this and all these diagrams.

Correspondingly, you have a particle propagating, but that particle can interact with the virtual particle, its own virtual particle. OK, and so the rule corresponding to the particle loop, it's-- so this is the real external particle, but anything coming in the loop, you can imagine how the virtual particle which come out from the vacuum, and then this can be interpreted as the particle interacting with its own virtual particle, yes, a virtual particle coming out from the back. Yeah. And so that, this kind of interaction, will affect the property of the propagation but can have the most effect by prefactor, but actually can change the mass, too.

But for the-- oh, it can correct for the mass-- and also, that's the subject of the QFT2-- can correct for the mass, I think at most can change the overall factor by z. Other questions? Good. Good. OK.

Good. If you don't have other questions, so let's conclude our discussion of chapter 3 on interacting theories. So as I said before, as I mentioned before, and now you are really equipped with the technique, now in principle you can treat any interacting theory. So the technique, even though we just used the scalar theory, but the technique is the same. OK, just for different theory, you have different details, OK, and now you have already equipped the foundation, the basic tools for dealing with any interacting theory.

And for the goal of this course, we want in the end to be able to calculate the, say, interactions in quantum electrodynamics, OK, and for that purpose, we still need to have some other preparations. So now let's discuss how to describe fermions. OK, and now we've described the scalar and how the scalars interact, and now we'll talk about fermions.

OK. So let me say a little bit of history. So, soon after the quantum mechanics was proposed by Schrodinger and Heisenberg, et cetera, and then people tried to generalize to relativistic situation-- OK, so that came from this Klein-Gordon equation, which we discussed before, which in-- this was the first attempt to write down a wave equation for relativistic particles.

OK, and we discussed before, this does not really make sense as a relativistic quantum mechanics. And yeah, yeah, but at the time, this Klein-Gordon equation, if you interpret it as a wave equation, suffers some-- suffers from some difficulties. One is that it does not have a positive probability. You cannot define-- I should say cannot define positive definite probability. And the second difficulty is that it has negative energy state. OK.

So yeah, of course, as I mentioned before, that this was-- there's more fundamental reason that actually relativistic quantum mechanics does not make sense, but at the time, in the late 1920s, people didn't realize that. OK, people just looked at those difficulties, they thought it's a technical difficulty. So Dirac proceeded trying to correct those difficulties, to overcome those difficulties.

OK, so Dirac then soon came up-- so I think this is around 1926, and then 1928, then Dirac came up with this Dirac theory, this Dirac equation. OK, so Dirac equation was aimed to cure those problems, OK, so Dirac concluded that the Klein-Gordon equation, the reason it had those problems was because the Klein-Gordon equation has a second-order derivative. And then he said, if we have a first-order-- have an equation with first-order-- yeah, he speculated, if we had the equation with first-order-in-time derivative, just like the Schrodinger equation, and then both problems maybe can be solved, OK, can be solved. And then Dirac came up with the Dirac equation.

So it turns out that the Dirac equation solved the first problem, OK, but didn't really solve the second problem. OK, but the level is, again, due to more fundamental reasons, you cannot really interpret the Dirac equation as a relativistic quantum mechanics equation, a wave equation for relativistic quantum mechanics. Actually, the Dirac equation should be interpreted as a field theory. So nowadays, we interpret this this is the-- gives the field theory for--

OK, so of course, Dirac didn't know this, so essentially, he discovered this beautiful theory for the wrong motivation, OK, for the wrong motivation. And yeah, this happens over and over again in physics. OK, people make great-- made great discoveries often for the wrong motivations.

But the key is that if you are good enough, you will find something new and that something new will be useful. [LAUGHS] OK, and this Dirac theory is a prime example which is actually one of the most beautiful-- we will see, this is one of the most beautiful equations in mathematical physics. Yeah, but also, it's actually describing electrons, so it's not only beautiful, but it's actually useful.

OK, so this, first, I'll talk about, introduce the Dirac equation, OK, and its covariance. OK, so the best way to introduce the Dirac equation was still his original motivation, is that we want to find the first-order equation which is Lorentz invariant. OK, so the goal is that you write down the equation, like a Schrodinger equation, which is first-order-in-time derivative, OK, but is Lorentz covariant. OK, but it's Lorentz covariant. OK.

So but Lorentz covariant means that this equation has the same form when you go to a different frame. OK, yeah, that's what we mean by-- just when you go to a different Lorentz frame, the equation, the form of the equation looks the same. Just different observers in different laboratory, they see the same equation, OK, so that's what we mean by Lorentz covariant.

OK, so but for this to be Lorentz covariant, remember, Lorentz transformation transform t to x, so immediately, you conclude that H must be first-order in spatial derivatives. OK. So the only-- then the most general way you can write it, so let's try to write the most general-- yeah. Yeah, let's try to write something like that.

So your first-order derivative, then, has to have the following form, alpha minus i-- yeah, so this is the gradient operator, yeah, spatial derivatives, and yeah, that's i just for convenience. And then this is a vector, have three components, and then have to contract with something, and then we include the alpha. And then you can at most add the constant, OK, so let me just write this-- for historical reasons, write this constant as m times beta. OK.

So if you look at this equation form, you say this doesn't make any sense, OK? So alpha and beta, they have to be some kind of constant, OK, but if alpha and beta are constant, then this is not even a rotational invariant, not to mention Lorentz environment, OK, because this derivative is not contracted by any other Lorentz indices. So alpha is just some constant, OK, so this cannot be Lorentz covariant, even cannot be rotational environment, OK, if alpha or beta are constant. OK, and psi is the ordinary function. OK.

So yeah, so you can have alpha x, partial x, alpha y, partial y, alpha z, partial z, OK, so you can easily convince yourself, when you rotate xyz, this is not symmetric because alpha is some constant. And so I'm sure this idea came to many people trying to look for some first-order equation which are Lorentz covariant, then after five minutes, you realize this is not possible. OK, this is just so simple, it's just not possible.

OK. But then Dirac made it work. OK, it's really, say, a stroke of genius. It's really a stroke of genius because there was nothing like this before.

Just even from a mathematical point of view, it's purely, purely imaginative, OK, just nothing like this before. Like when Einstein wrote down his theory, et cetera, you can still trace-- there some clues, OK, but when Dirac-- this one just really--

[LAUGHTER]

--like music, just came out from his mind, OK? It just-- and then he reasoned, OK, if this is constant which does not work, and then let's make alpha and beta to be matrices. Take them-- OK, and then, so let's say they are n-by-n matrices. Then in order for that equation to make sense, then psi has to be a n-component vector.

OK. So even for some people, you come up with this idea, you will not imagine this will work. OK, you just say, oh, this will be a mess, but somehow, he made it work. OK, so we will see how to make this work.

And so now if you want H to be Hermitian, and then you can immediately conclude-- so that's why I put the minus i thing here, is the alpha and the beta. So m will just be some constant, OK, and here is a matrix. You can always take a constant out, OK, so alpha, beta are Hermitian matrices. They're just some constant Hermitian matrices.

And then, so then he reasoned that for this equation, if we want this equation to be Lorentz covariant, then at least it should have the relativistic plane wave as its solution. OK, if you have a covariant equation, and at least you should have the relativistic plane wave as its solution. If it does not even have that type of solution, then you, yeah, of course cannot be covariant.

OK, so the minimal requirement-- so before we really try to see how can we make this into a covariant equation, we say let's consider minimal requirement. So we'd like to-- here, let me call this equation star. Star should have plane wave solutions with the standard relativistic dispersion relation, OK?

Now, these, you should have p squared equal to minus m squared. OK, you have a plane wave. The plane wave will be labeled by p, and then you should have p squared equal to minus m squared. Then the m will be its mass, OK, its mass.

And to do this, the simplest way to do this, so we know that the Klein-Gordon equation-- so we know the Klein-Gordon equation have such solutions. OK, Klein-Gordon equation have such solutions. And then the simplest way to do this is-- this is the first-order equation.

And now imagine if we square this equation, and then this should reduce to the Klein-Gordon equation. If you can make that work, then this property will be satisfied, OK? So this will be satisfied-- can be satisfied if square of star, OK, satisfies reduced to the Klein-Gordon equation.

OK, so now let's try to do this. So when we square this star, we just act twice, so essentially, you have-- so you have-- so when you square that equation, then you just get partial partial t square psi equal to H square psi. OK, and then we try to make this of this Klein-Gordon form.

OK, so the right-hand side, we just have the form minus i with alpha dot with this, and then plus beta m square psi. OK, we have these, and then you can just expand this explicitly, so the right-hand side, so then we have minus-- let's square this first-- then you have minus alpha i alpha j.

OK. And then you have partial partial xi partial partial xj psi, and then let's square this. Then you have beta square m square psi, and then you have cross term. So cross term now has the form beta alpha i plus alpha i beta.

But now remember, beta and alpha, they are not constant. They are matrices, OK, so they don't necessarily commute. So you have to be careful about the orders. OK, and m partial psi partial xi. OK, so the right-hand side is just like this when you square it.

And now we wanted to look at the Klein-Gordon equation. The Klein-Gordon equation has the following form. So now we want it to be-- OK, hopefully to be given by minus partial x square psi plus m square psi. OK, so this one would be the Klein-Gordon equation, OK, which we wrote before. OK, so by x-- yeah, here, sorry, I should say xi square. Yeah. OK? So i should be summed, OK?

So now we want this equation, so we want the right-hand side to be equal to this equation. If that works, then we're guaranteed to have a plane-wave solution of such a dispersion relation because this equation has-- OK, because this equation has the plane-wave equation with-- yeah, plane-wave solution with that kind of dispersion relation.

So now we just compare the both sides, so for this to be true, so we just need to have-- so let's compare the second-order derivative term. And then we need alpha i, so we need-- first, when i not equal to j, the off-diagonal terms, they all should vanish, OK, because here there's only diagonal terms. So that means that the alpha i alpha j plus alpha j alpha i should equal to 0.

OK, remember, the matrices, they don't commute, so we should be careful about the ordering. OK, so when i not equal to j and when i equal to j, then you should just reduce to here. OK, and that means that the alpha i square should equal to-- and here you should just-- here you should just-- this term just will give that, and the-- should equal to beta square equal to 1. OK, so here, there's no summation, OK, no summation over i.

OK, so each alpha i square has to be 0, and now we want this linear term to be 0. And then you want beta alpha i plus alpha i beta to be 0, OK? And also, just let me put it together. We said the alpha and the beta, they must be Hermitian, so that means that the alpha i dagger is equal to alpha i and the beta dagger equal to beta.

So if we satisfy all these four conditions, and then we will guarantee that that equation star should have the plane-wave solution. OK, that should have the plane-wave solution. And so now you just try to find matrices, satisfy those conditions. OK. So we already said that the 1-by-1 matrix don't work, that if they are constant, of course, they don't work.

And so you can also check 2-by-2 matrices. It's not enough to do this. 3-by-3 is not enough. When you go to 4-by-4, then you finally find the solutions. OK, you actually find the infinite number of solutions.

So you see that to satisfy them needs at least a 4-by-4 matrix, so n has to be 4. OK, so this n has to be 4. And so let me give you some possible solutions.

So for example, here is one solution. Say you take beta to be 1, so all matrices here should be understood as 2-by-2 matrices, OK, so all together, it's 4-by-4 matrix, so 1 and minus 2. So beta's given by this, so this is one solution, and alpha i given by 0, sigma i, sigma i, 0. And sigma are Pauli matrices, OK, sigma i, Pauli matrices.

OK, so this is one solution you can check yourself. OK, I will not do it here. And this actually is a solution that satisfies all these four conditions, and here is another solution.

So now let me just save time in not writing down this two-by-two. So for example, beta can be 0, 1, 1, 0. OK, this one here is a two-by-two matrix, OK, and alpha i equal to sigma i, 0, 0, minus sigma i. OK. So you can check both of them satisfy those conditions. Yes?

AUDIENCE: Sorry, so when you have alpha times grad psi, is it-- do you act the grad on each element of psi and then multiply by alpha? Or do you-- like, what's the order of operations?

HONG LIU: Oh, it doesn't matter because alpha is just some constant, right? Alpha and the grad, they don't act on the same space. So that derivative just acts on derivatives, and alpha acts on a different component of psi.

AUDIENCE: I see, but if you took the grad of psi and then multiplied by alpha, then you'd mix up the components of it.

HONG LIU: Yeah. Yeah, it's fine. Yeah. Yeah, you can do it either before-- yeah, they commute, so the operation of the alpha and the operation of grad, they commute. Yeah. Yeah?

AUDIENCE: Yeah. So you'd have a-- so alpha x because alpha x would just be the Pauli x matrices times the grad x, and then you'd need another one for y and for z?

HONG LIU: Yeah. Yeah. Yeah. So yeah, so I urge you-- so here I don't have time to write everything very explicitly. So here you just write it as alpha x partial x plus alpha y partial y plus alpha z partial z, and each alpha is a matrix. And alpha acts on different components of psi, and this just acts on-- the derivative acts on all components of psi. Yeah.

OK? Good? So now, good, you say we have an equation, OK, so far, so psi-- so alpha is a 4-by-4 matrix. Yeah, alpha and beta are 4-by-4 matrices, so that means that psi should be a four-component vector.

OK, it's a four-component vector, OK, so we will take-- so we will denote it as psi alpha. So alpha equal to 1, 2, 3, 4, OK, and we-- for the moment, let's just take the most general situation. We'll take alpha to be complex, OK, each of them to be complex. OK. Yes?

AUDIENCE: Is alpha, like, a vector of matrices?

HONG LIU: Yeah.

AUDIENCE: Which element of alpha is a matrix?

HONG LIU: What do you mean?

AUDIENCE: So, like, alpha i is a matrix.

HONG LIU: Yeah, alpha 1, alpha 2, alpha 3, they are all-- they are three matrices.

AUDIENCE: And you said we'd need the first on the list. Wouldn't we want each alpha i to be diagonal? Is that what you're saying, that each matrix in alpha should be a diagonal matrix?

HONG LIU: No. No, no, no, no, no, no, no. No, alpha, no, we don't know the form of the alpha, right? No, alpha are just matrices, so this means that alpha 1 alpha 2 plus alpha 2 alpha 1, as a matrix product, should give you zero. Yeah. Yeah, alpha itself is a matrix.

AUDIENCE: Of matrices.

HONG LIU: Hm?

AUDIENCE: Of matrices.

HONG LIU: Sorry?

AUDIENCE: Each alpha i-- each component of alpha is a matrix. Right?

HONG LIU: Yeah. Yeah, yeah, yeah. Yeah, just, alpha have three components.

AUDIENCE: Right.

HONG LIU: And each component is a matrix. Each component is a matrix. Just, if you have alpha 1, alpha 2, alpha 3, so here, here you see explicitly, alpha 1 is equal to sigma 1, sigma 1, alpha 2 is sigma 2, sigma 2, et cetera. Other questions?

OK, yeah, at the beginning, this may a little bit more-- not very intuitive, OK, but if you just work through it, then you will get a feeling about it, OK, you will get a feeling about it. So that's why I say this was really genius, because just nobody could have thought of this. OK, it just came from nowhere. Really, there was no clue, OK? There was no clue of such a structure.

Yeah. OK, so this is a new object, so we call it spinor. OK, we call it spinor because it-- later we will see that this describes spin-half particles, so that's why we call it spinor. OK. Good.

AUDIENCE: Dr. Liu?

HONG LIU: Yeah?

AUDIENCE: So if we take n to be larger, then we describe all the spins?

HONG LIU: Hm?

AUDIENCE: If we take a n by n matrix where n is not so small, then we can use that --?

HONG LIU: No, you get-- just we are not using those matrices in the efficient way. Yeah. Yeah, this become-- you can reduce always-- yeah, just from physical purpose, you can always reduce it to 4, yeah. Yes?

AUDIENCE: Well, what if I wanted a wave equation for higher spin?

HONG LIU: For higher spin?

AUDIENCE: Yeah, like 3/2 or 5/2.

HONG LIU: Yeah. Yeah, if you know how to do the two halves, then you can generalize. Yeah. Yeah, so one half, essentially, you can-- yeah, based on one half, you can generalize it. Yeah.

AUDIENCE: Thank you.

HONG LIU: Yes?

AUDIENCE: Just to clarify, so alpha i, is it a 2-by-2 matrix of 2-by-2 matrices? Or is it just a 4-by-4 matrix and that's just a convenient way to write it?

HONG LIU: Yeah, this is a 4-by-4 matrix. It's just a convenient way to write these 4-by-4 matrices.

AUDIENCE: Sure. OK.

HONG LIU: So that I don't have to write all four components. I just-- yeah.

AUDIENCE: Yeah, yeah, yeah. OK.

HONG LIU: So I divided these 4-by-4 matrices into four 2-by-2 blocks, and then I specify each block.

AUDIENCE: Yeah, it's just blocks, not matrices in the matrix.

HONG LIU: Yeah, yeah. No, no, just the blocks of that 4-by-4 matrix, yeah, separate the single 4-by-4 matrix into four 2-by-2 blocks.

AUDIENCE: Yeah.

HONG LIU: OK? Good? OK, so for later convenience, let's introduce a slightly different notation. OK, so now we have this equation, so now we have the form of this. So now we have the form of this partial t psi equal to minus alpha plus beta m psi, so now let's multiply both sides by a factor-- by beta. OK, so beta is a matrix, OK, so this is a matrix equation.

OK, so this is a matrix equation. Let's multiply both sides by beta, OK, and then we get i beta partial t psi equal to minus i beta times alpha. Yeah, and let me just write it maybe this way, so alpha i and partial xi psi, OK, then plus m psi. OK, so you'd get an equation like this.

OK. So now I will denote-- introduce a new notation so that it looks nicer. I'll denote the gamma 0 equal to i beta and then the gamma i equal to i times beta alpha i. OK, and then let's all pull it to the same side, and then this becomes the following equation, then the equation has the following form, gamma mu partial mu minus m psi equal to 0. OK, so this is the form of our Dirac equation.

OK, so this becomes gamma 0 times partial t, so this becomes gamma i times partial xi. When they come together, then become gamma mu partial mu, and then m will move to the other side. OK. So due to different conventions of the Minkowski metric signature you use, et cetera, so different books or different places, you will see i in different places. Some equations have i-- yeah, some books have i here. Some books have i here. In my version, there's no i, OK?

[LAUGHTER]

And so I know this is annoying, but yeah, but this is just a fact of life. Yeah. Yeah, so I find that this version is most simplest in terms of notation anyway, so that's the convention we use. Good. Good.

So now, again, this is a matrix equation. Now let me just write it in the component form, OK, so this is in the component form. I have gamma mu, which-- each of them is a matrix, alpha, beta, so alpha and beta, they're always run. Yeah, sorry for this.

Now the alpha, beta just means the-- yeah, means the indices, OK, not the matrices. So alpha and the beta-- actually, convention is to write this like that. OK, it doesn't matter.

So partial mu psi beta minus m psi alpha equal to 0, OK, so this is a matrix equation. There are all together four equations, so the beta is summed because beta is repeated. So beta is summed, and then, yeah, and the mu is also summed.

So this is a little bit intricate equation. OK, so this is a little bit intricate equation, but once you get used to it, it's not that difficult. Yes?

AUDIENCE: There's no meaning to the upstairsness or downstairsness of alpha and beta, right?

HONG LIU: There's a meaning of upstairs, yeah, because these two index are not symmetric, so it's easier to put one upstairs, one downstairs. Yeah, these two indices are not symmetric. Yeah. OK. So yeah, it takes a little bit time to get used to it, OK, and I know some people develop psychological fears for fermions because you have to deal with those gamma matrices, OK? For a long while, actually, I have this psychological fear myself.

[LAUGHTER]

When I look at fermions, I want to be away from it because I don't want to deal with those gamma matrices, but these are beautiful objects if you get used to them. OK, so now those conditions, you can also write them in the compact way in terms of the gamma-- in terms of gamma matrices. OK, so 1, 2, 3 now can be written as-- in terms of gamma matrices gamma mu gamma nu plus gamma nu gamma mu.

So mu nu is always from 0 to 3, OK, equal to 0, or mu not equal to nu, and then gamma 0 squared is equal to minus 1. It's because this gamma 0 is i beta, and the beta squared is equal to 1. And then the gamma i square, so there's no summation over i here, OK, because each matrix gamma i square equal to 1. OK.

And you can write this further in a more compact form. You can write this further in a more compact form. So you can write this more compactly as gamma mu gamma nu anticommutator equal to 2 eta mu nu. So anti-commutator just means that if you have objects, two objects, curly bracket means a b plus b a, OK? So this is the key equation.

OK, so all the gamma matrices, so you can check yourself, OK, all these are given by just this. You can easily see, when mu not equal to nu, of course, the right-hand side is 0, so that just is equal to that equation. When mu equal to nu, when there is-- when they're both 0, then this gives you minus 1. That's corresponding to the gamma 0 case, and then when they're ij, then corresponding to gamma x. OK.

And so this equation, when the later mathematicians, of course, studied this, a mathematician would say this is a beautiful object. And then they studied this, so now it's called Clifford algebra. So this object is called Clifford algebra.

OK. And so then any sets of gamma mu-- so gamma mu are a set of four matrices satisfying this equation-- OK, so let me call this star-- is called a representation of the algebra, of the Clifford algebra. OK. OK.

So from these two solutions of alpha and beta, we can easily-- from here, we can work out what is the gamma nu and the gamma i, so yeah, so here are two examples. Also, yeah, before-- talking about examples, also, from number 4, you also find that the gamma 0 dagger equal to minus gamma 0, and gamma-- so gamma 0 is anti-Hermitian and the gamma i is Hermitian. OK.

And you can also write this together in the more-- or equivalently write it as gamma mu dagger gamma 0 gamma mu gamma 0. OK, so you can check that this equation is the same as these two equations. OK. Yes?

AUDIENCE: Do these representations generate a certain transformation?

HONG LIU: Yeah, yeah. I will talk about things related to this a little bit later. Yeah. Good? And then we can write down explicit solutions for those gammas, so that's two representations.

From those solutions of beta and alpha, we can write down different solutions of gamma. So for example, for 1, you're corresponding to gamma 0 equal to minus i. So now this is minus i times a 2-by-2 matrix, OK, or i, 0, 0, minus i, OK, and the gamma i is equal to 0, minus sigma i, minus i sigma i, 0.

And the second solution there corresponding to gamma 0 equal to 0, i, i, 0, and then gamma i equal to 0-- I think it's also minus i sigma i, i sigma i, 0. OK. Good, so these are just-- again, these are just good-- so these are two different representations of this algebra. OK.

So now let me make some remarks. So before I proceed further, do you have questions? Yes?

AUDIENCE: For 2, shouldn't the Pauli matrices be-- oh, nevermind.

HONG LIU: Oh. Other questions? Yes?

AUDIENCE: So I guess that I don't understand what space this side vector, the four entries-- like, what is that? Is this a Lorentz four-vector?

HONG LIU: No, it's not a Lorentz four-vector.

AUDIENCE: So in what sense is it a vector? How does it transform?

HONG LIU: Yeah, so this is a new space. We will talk about that. Yeah. Yeah, so this is a new space, and so that's called-- this is called spinor space, yeah. Yes?

AUDIENCE: How do we know that-- are these the only two representations?

HONG LIU: Oh, no, no, no. There are infinite number of them. Yeah, I mentioned there are infinite number of such solutions, and this is just two of them. Yeah, I will comment on all those different solutions. Other questions?

OK? Good? So now let me make some remarks. OK, so first, if you consider the case m equal to 0, OK, if you can see the case m equal to 0, and then this is, like, for massless, so when m equal to zero, then when you reduce to this Klein-Gordon equation, what you get is the massless equation.

OK. OK, it's a massless equation, and then you will have dispersion relation p squared equal to 0. OK, so in this case, the original equation just becomes partial t equal to minus i alpha i partial partial xi psi. OK, so that's your equation. There's no this m beta term. There's no m beta term.

So in this case, you just-- the same story just here, you just forget about beta. OK, the same story, you forget the beta, and then you only need alpha i alpha j the commutator to be 0 for i not equal to j, essentially just that equation 1 there. And then also, you'd want the alpha i square equal to 1, OK, so for any i. OK, so now these are the conditions for the alphas, and now you can actually satisfy by 2-by-2 matrices. So what matrix satisfies this kind of relation?

AUDIENCE: Pauli.

HONG LIU: Yeah, so the Pauli matrices anticommute among themselves, and their square is equal to 1. So you can just take off i in order to simplify here. So this tells you something important, tells you actually the equation for a massless particle and a massive particle are very different.

So massive particle, you actually need the four components, but for the massive-- but for the massless particle, you can describe by a sigma matrix here, only 2 by 2. OK, so that means for massless psi, it can be described using a two-component vector, OK, so that in this case, psi equal to psi 1 and psi 2. Again, they are all just complex. OK.

Good? So this actually-- yeah, we will elaborate on this later. So this actually tells you that actually the massive particle have more degrees of freedom than the massless particle, OK, more degrees than the massless particle. Good? And then the second thing is what Dirac essentially was-- Dirac's original motivation.

So from Dirac equation, you can show you can derive a current. Just as you can do for the Schrodinger equation, you can derive an equation like this for some j mu and with j 0, with the 0-th component of this thing, positive definite classically. So I emphasize that this is classically. You will-- later, you will see why, OK?

And this, I will leave it to your pset, OK, so this is very similar to the derivation of such occurrence in the case of the just non-relativistic Schrodinger equation because this has the same structure. Yeah, the Dirac equation, when we start it, has the structure of the non-relativistic Schrodinger equation, and yeah, so you can show something like this exists. OK.

And then the third point is related to the question many of you may have. So we said, what's the meaning of all these different solutions for alpha and beta or for gammas, OK? So as I mentioned, you can have infinite number of solutions. What's the meaning of them?

So first, let's imagine when we look at this equation-- so as I said, this is a matrix equation, so in this matrix equation, then you have this psi which is a four-component vector, OK, some four-component vector. So now let's imagine we make a basis change in this four-component vector, OK, so imagine, just say, consider making a basis change in psi.

OK, so a basis change in psi just means, in linear algebra, this is a vector, means we consider another psi prime, which is-- so we take psi to some other psi prime which is related to psi by invertible matrix, so B, just some constant, complex, invertible matrix. OK, so essentially, you just make a linear superposition of different components. OK, so yeah, so this corresponding to a basis change.

So now if psi satisfies this equation, psi satisfies this equation, you can easily convince yourself that the psi prime satisfies the following equation, the gamma mu prime partial mu minus m psi prime equal to 0, and the gamma mu prime is equal to B gamma mu B minus 1. OK. So yeah, so this easily can be shown.

You just multiply B from both-- just multiply B to this equation, OK, multiply B to this equation, and for this term, you can just directly give-- psi gives you psi prime. And for this term, then you just get the B, and then here you can insert B minus 1, B, and then, yeah, you get that. OK, so this, you can easily convince yourself, just a couple of lines.

So now, so psi prime satisfies essentially the same equation but a different gamma matrix, OK, a gamma mu prime, so now you can easily check yourself, OK? So you can easily check yourself this gamma mu prime also satisfies that algebra. OK, so you can easily check yourself.

So then we conclude, any sets, since we can make a basis transformation as you want, OK, that the-- except the basis transformation should not change physics, OK? So any sets of gamma mu related by a similarity transformation are equivalent. OK. OK, they're equivalent, and because they should not give you new physics just because one is going to change the basis.

OK. So now, here is a highly nontrivial mathematical statement, so now-- which of course I will not prove here because it'd just take too much-- yeah. So you can show-- I'll just quote the result. So you can show, OK, with a little bit of effort that under such kind of equivalence relation, means that the similarity transformations, they are all equivalent, and such equivalence relation, the representations of star is unique.

OK. So you can show any matrices which satisfy that equation, they're all related by similarity transformation, OK, so they're all physically equivalent. They're just corresponding to a change of basis. OK, they're corresponding to a change of basis.

OK. So but different forms of the gamma, they may-- different forms of the gamma matrices, they may be useful for different purposes. OK, they may be useful for different purposes. For example, this I, solution I we wrote down before, it's convenient for if you want to take the nonrelativistic limit. For example, if you want to make a connection with a nonrelativistic quantum mechanics, and that's actually the most convenient form, the matrix to use.

And II is actually in the-- opposite regime, it's convenient for the ultrarelativistic regime. OK, so it depends on which regime, sometimes you use different gamma matrices. OK, they make your algebra a bit more convenient.

So now having introduced the Dirac equation and then the structure of the Dirac equation, but still we haven't showed that the Dirac equation is covariant. OK, we just showed that the Dirac equation can have the plane-wave solution and that the plane-wave solution will have a standard of nonrelativistic dispersion relation. OK, so in order to show that the Dirac equation is covariant, we have to show-- we have to make a Lorentz transform and show that the Dirac equations are the same in every Lorentz frame.

OK, we have to show that, OK, and we are running out of time. So we, of course, won't have time to do that today, but let me just remind you how this Lorentz covariance works for the scalar case. OK, so remember, so recall, yeah, let's just quickly recall, for scalar, we have phi x.

Then under Lorentz transformation-- so consider Lorentz transformation which x mu goes to x prime mu equal to lambda nu mu x nu. Consider such a Lorentz transformation, OK, so and then phi transforms as following, phi prime x prime phi prime. New phi evaluated at the new position should be the same as phi evaluated at the old position, OK? Or let me just write it here so that I don't have to erase. So the phi-- yeah, just equal to phi lambda minus 1 x. OK.

So now if you look at the Klein-Gordon equation, let's see how this is covariant, OK, so now let's see in a different frame. OK. So covariance means that when we go to a new frame, partial prime square-- OK, so this means in the prime coordinates-- minus m square and phi prime evaluated at the x prime, there must be a-- yeah, so this is in another Lorentz frame, OK, so they have the same form. Indeed, you see these two are equivalent because this just equal to that.

OK, trivially, you could do that just by definitions. And this is a Lorentz scalar, so this is also equal to that. OK, so you see that the Klein-Gordon equation indeed is Lorentz covariant. OK, it's the same in any frame.

So now we want to show, OK, so now we want to show that the Dirac equation has the same property, OK, and that is much more nontrivial. OK, that's much more nontrivial. Again, it's really ingenious, ingenious, yeah, but we see, actually, it works. OK, so we will do it next time.