[SQUEAKING]

[RUSTLING]

[CLICKING]

PROFESSOR: So first, I'll try to remind you what we did at the end of last lecture which was last Wednesday. And so we introduced these things called the Grassmann variables which are anticommuting. Classically, they anticommute-- so for example, if you have, say, a quantity called theta, and then theta 0-- theta squared equal to 0.

And then two such kind of Grassmann variables commute with-- anticommute with each other. And then if you consider a function of such theta, then you just do a Taylor expansion, and you only have two terms. So the first term is a constant, and the second term is proportional to theta, because all higher power terms vanish.

So the same thing with a multiple variable-- just expand until you reach the square of any variable. And then you can do a differentiation. So we always define the differentiation from the left.

So this is given by f1. And we can also define the integration, and the integration is determined by two rules. The first rule is d theta, a constant equal to 0, and, also, d theta theta equal to 1. OK, so based on these two rules--and then you can just work out the integral of any functions.

So before I proceed, do you have any questions on this? Good. So for multivariable function, you just expand.

Say, for example, if you have two variables, then you just have f0 plus f1 theta 1 plus f2 theta 2 plus f1,2 theta 1 theta 2. So we can also define the integra-- yeah, the differentiation of a multivariable function-- more easy. You just be a little bit careful of the direction you take the derivative.

So now, let's look at the example of the integration. OK, let's look at the integral of a function of two variables. So you should keep in mind that this order is important because theta 1 and theta 2 are anticommute.

So if we write d theta 1 and d theta 2, then it is d theta 1 d theta 2. It's equal to minus d theta 2 d theta 1. You want to change the order.

So we can just do the integration by using this rule. You just plug in this expansion into here, and then, obviously, all these three will give you 0 because of the-- and only the last term will contribute. And then you just get f1,2, then d theta 1 d theta 2, then theta 1 times theta 2.

So now if you want to do the integration, because theta 2 is before the theta here. Theta 2 is closer, and the theta 2 before theta 1-- and then you need to change the order. So we can do it by doing f1,2 d theta 1, and then you have d theta 2 integral, theta 2, and then theta 1.

Now we have changed the order of theta 1 theta 2, and this gives me 1. So after this, it gives me 1. This also gives me 1. So this is just equal to minus f1,2.

So similarly, you can just do an arbitrary number of integrals with-- you can just do integrals with an arbitrary number of variables. Just keep in mind that d theta 1 d theta 2 theta 2 theta 1-- this is equal to 1. So you do this first, and then you do that, and then you get the right order.

Any questions on this? Yes.

AUDIENCE: Here, f0 and f1, are those complex numbers, or Grassmann numbers, or both?

PROFESSOR: Here, we just take them to be ordinary numbers. So they can just be complex numbers. That's right, yeah. Other questions? OK, good.

So now we can look at a little bit more complicated integral. So now let's look at the Gaussian. So let's look at such a Gaussian integral.

So this Gaussian integral looks complicated, but, of course, it's simple. Because again, this is when you expand it. There's only a single term left.

So this is just equal to d theta 1 d theta 2, and then 1 minus this guy. Yeah, so theta 1 a1,2 theta 2. Because the next term, when you do the Taylor expansion, will give you 0. A little bit. So that will involve the theta 1 square or theta 2 square. And now if you use this rule, you just get a1,2.

And so you can now generalize two Gaussian integrals with an arbitrary number of variables. And so we can write a more general integral, d theta 1 d theta 2 n, because they-- so minus 1/2 theta i a ij theta j. So now here, you should assume i and j are summed. OK, so ij is from 1 to 2n.

So for this integral, again, the strategy is simple. You just, again, expand it and then just do term by term. So at a certain time when you expand to a certain order, the theta will repeat and then the expansion truncates. But you can actually easily convince yourself without doing any calculation to see what the answer is.

So if you don't expand to a sufficient number of theta, you will get 0 because you have to have each theta for d theta in order for this rule. So you have to have one theta for each d theta. So that means you have to expand to the order which each theta just appears exactly once. And then if you do that-- if you look at that term, you find that this just gives you square root determinant aij determinant a.

You find this actually gives you-- here, a is an antisymmetric matrix-- by definition, it should be anti-- yeah, because theta i and theta j are anticommute. So a ij should be an antisymmetric matrix. OK, any questions on this?

So this looks like standard Gaussian integral. Remember the standard Gaussian integral. What would you get? If you have such kind of quadratic structure for the standard Gaussian integral, for ordinary numbers, do you remember what you get? Yes.

AUDIENCE: Yeah, it's like 2 pi to some power over [INAUDIBLE].

PROFESSOR: Exactly. Right. So if theta are ordinary variables, then you will get some constant divided by square root of determinant a. But for the Grassmann number, you just solve it, and you get the proportional to the determinant-- not divided by the determinant.

Good, so we can also introduce complex Grassmann variables. So, for example, I can introduce theta over the square root 2, theta 1 plus i theta 2. So theta 1 and theta 2 are considered to be 2 real Grassmann variables.

i is just ordinary i, and then the theta star would be, of course, the theta square root 2 theta 1 minus i theta 2. And so the rule for defining the product-- the complex conjugation for the product-- is defined to be eta star theta star. So this is different from the ordinary variables. So this is defined more like for an operator.

So for Grassmann variables, when we define the complex conjugate, we actually change the order-- as well, what we normally do for a Hermitian conjugate. So now, if you have a function, say, theta theta star, with theta now a complex of Grassmann variables, you just do the same thing. So you have c0 plus c1 theta plus c1 bar theta star, then plus c1,1 theta theta star. So again, you're just expanding theta and theta star. And then you truncate here, because the further terms will involve either theta square or theta star square.

So here, for the integration, we define d theta d theta star and then theta star theta to be 1. So we choose this convention. So we define this to be what? So that's corresponding to a specific choice of a measure for this d theta d theta star.

Good? So now we can look at the complex Gaussian integral. OK, now let's look at Gaussian.

So if you have d theta star d theta, say, exponential minus theta star b theta-- so b is just some number-- some arbitrary number. And again, you can just do it by expanding this in power series. And just as in that example, you just find that this is equal to b.

And you can also now do multiple variables-- so multiple variable Gaussian. So suppose you have now j equal to 1 to n d theta j star d theta j-- the product of all of them. So if you have minus theta i star A ij theta j-- again, i and the j should be assumed to be summed.

And then when you expand it, again, you expand precisely to the order-- only one term in the expansion contribute. It's the expansion in which each theta and theta star appear exactly once. So when you look at that term, you find that this precisely gives you det A.

So again, this, should be contrasted with the complex integral-- ordinary complex Gaussian integral. So if you have an ordinary Gaussian integral for complex variables, again, you will get 1 over determinant A. It's some constant over determinant A. So for Grassmann, you just get determinant A. And this feature is very key in distinguishing fermions and the bosons, and plays a very important role.

So we can also consider more general integrals, more general Gaussian integrals like this. So now, let me just introduce a little bit of notation. So let me call the theta equal to theta 1 theta, say, n. Let me introduce eta. Eta is some other Grassmann variables, eta 1 eta n.

And then we can consider an integral like this which we will encounter. Later, we will encounter integrals like this. Again, d theta star j d theta j-- then you have exponential, so minus theta dagger.

So now, the dagger is including both the transpose and taking the star. And A, now, is just the matrix. Again, just write this in the matrix notation.

But now, suppose that here I only have a quadratic term. But suppose now I have some linear term-- say, eta dagger theta and theta dagger eta. OK, suppose now I have this. And again, you can just complete the square through the Grassmann version of the complete-- the square. And then you can convince yourself that this is given by det A-- determinant of A-- matrix A-- and then exponential eta dagger A minus 1 eta.

So up to the sign and i, et cetera-- this is exactly what you'd normally expect after you complete the square. You get A minus 1 here, and then you get this eta. Any questions on this? Yes.

AUDIENCE: Is A still antisymmetric?

PROFESSOR: It does not have to be antisymmetric. Yeah. So it can be some complex matrix, in principle. Yeah-- some general complex matrix. So I will also denote this by I eta and eta star.

OK, so we only need the one last formula, and then we can talk about the path integral for the Dirac fields. So one last formula is that we, say-- let's try to calculate the following integral-- 1 over I0,0. OK, I0,0 means just this integral. Just this integral is I0,0 with both eta equal to zero.

So 1 over I0,0-- suppose let's consider this integral, j from 1 to n d theta j star d theta j. Now suppose we have theta k star-- theta k-- and theta l star in downstairs, and then you have the exponential minus theta dagger A theta. Suppose you have an integral like this.

So the integral like this can be obtained from this one by taking derivatives. So this is like a generating functional. So then this can be considered as 1 over-- yeah, let me just suppress these two zeros-- 1 over I. This can have minus partial partial eta star k, and then yeah-- partial partial l. OK, then take derivative on I eta and eta star-- taking derivative on this, and then after you take the derivative on each side of the eta equal to eta star equal to 0.

So when you take these two derivatives, you can see that we bring down the theta k and theta l. And this sign-- just make sure you get the sign correctly. And then when you do this derivative here-- and then you find that this is given by A minus 1 kl.

So this is actually the same as the bosonic case. Remember, in the bosonic case, if you have a Gaussian integral, and if you have the two variables in the downstairs, then that gives you the inverse of the matrix up to a constant. Yeah, so this aspect is actually similar to the bosonic case. So do you have any questions? Yes.

AUDIENCE: Sorry if I missed it, but is there a name for that i?

PROFESSOR: Just the integral. Yeah.

AUDIENCE: OK, it's just that integral?

PROFESSOR: Yeah. Yeah, just a shorthand notation. Yeah, you can call it the generating function.

Yeah, so this is a generating function for arbitrary powers of this kind of integral, because any powers of theta-- any combination of theta k, et cetera, you can just obtain by taking derivative on this I. Yeah, mm-hmm. Yes.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah, you can-- here, the complex conjugate-- in some sense, they are independent of each other. Yeah, they are independent variables. Theta and theta star, because they depend on two variables-- and you can just view them as independent variables. OK, good.

So now, finally, we have the preparation to do the path integral for the Dirac fields. So remember, Dirac fields is a vector-- a spinor field with four components. There's four components, and they depend on space-time coordinates.

So normally, we just say this is the ordinary functions. But now, in order to correctly capture the fermionic nature of the Dirac fields, I now require that psi alpha x. So alpha is equal to 1, 2, 3, 4. Take values as Grassmann numbers.

So what this means-- so x is still our ordinary coordinate. For any choice of space-time location, x-- so this psi alpha gives give you a Grassmann number. And just for any choice of x, this just means that. And this is a function from an ordinary space-time variable to the space of the Grassmann numbers.

And in particular, this means that psi alpha x psi beta y-- they anticommute. So this is the rule. So now, that's the only difference. And now we can just do the path integral.

So the path integral is just completely parallel as before. So otherwise, exactly the same as before for what we did before for a scalar field. So the only difference is now the things are integrating. So in particular, we can write any correlation functions.

So, for example, again, let's use omega to denote the vacuum state. And the time-ordered correlation function-- the x denotes some product of operators. Yeah, our previous notation-- and the omega-- and divide it by omega. So this has a path integral in the present description in terms of psi bar D psi, and then x in the integrand, and then exponential i S. And then so everything is exactly the same as before, except the variables that you are integrating now are Grassmann variables, and the S is just whatever your action is.

So let me explain a little bit this notation, psi. So D psi just means you have, first, you have alpha from 1 to 4, and then you have D psi alpha t and x. And this notation is exactly the same as before.

When I write D psi, you should imagine I take the product of all components. And then for each of them, it's just exactly the same as we do for the bosonic field definition. The only difference is that now this is Grassmann variables. So yeah, also, the x here is, say, some product of psi's-- so arbitrary product of psis-- then everything previous takes over. You can just write down the expression.

So again, in order to do calculations-- for example, to calculate the propagator, et cetera-- it's convenient to use the generating functional as we do here for this Gaussian integral. So it's convenient to introduce this generating functional eta eta bar. So this is defined to be the D psi bar D psi.

And then you have exponential i s, and then you write, say, eta bar psi psi bar eta. So again, when you take derivative of eta, then you can bring down the powers of psi of psi bar. And then you use as a trick if you know this one-- and then you can just calculate any x.

Any questions on this? Yes.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Oh. No, you don't need to-- you can view this as a definition. We just carry over what you used as the definition. We carry over what we did before, and then we just replace everything by Grassmann variables.

Any questions? Any other questions? Good.

OK, so let's just try to calculate this in free theory. So in the free theory, we will eventually consider the-- so far, we just consider the Dirac theory. So let's just first consider Dirac theory.

So this is general. This is completely general. S can be anything, but for the Dirac theory, we have S0 is equal to minus i. So for the Dirac theory, it's simple, because it's just quadratic in psi, and then we just have a Gaussian integral here.

So the first thing to do is to write, again, this as a form of a matrix. So we have already done this before. So this has just become i d4x d4y, and then you have psi bar alpha x A alpha beta x minus y psi beta y. So this A alpha beta x minus y-- this is a matrix both in the spinor space and in the space of ordinary function-- in the space of functions.

So this is x minus y slash y. Yeah, so this means you take derivative on y. So once you write it in this form-- and then, again, we have this just in the Gaussian form.

So now this path integral is just in the Gaussian form. And, again, we can just generalize what we did before. Just keep in mind that those integrals-- now this is given by determinant A, and then we just find that the-- so, for example, now this Z eta-- Z0.

So this means that we can see the Dirac theory. So this just can be directly evaluated, and then this is given by determinant A. So now this determinant should be understood as both in the spinor space and in the space of functions.

Yeah, and then we have the analog of this term, and then we have exponential eta bar, then dot D. So I denote the A minus 1 as D in the motivation. And then the D-- so this should be considered as a shorthand notation for this.

So this should be considered as-- let me just write it here. So this should be considered as a shorthand notation for d4x d4y eta bar alpha x D0 alpha beta x minus y eta beta y. So this thing is a shorthand notation for this.

And this D alpha beta 0 x minus y is just the inverse of this matrix A. You just should view it as an inverse of the matrix A. And that's the same thing as what we did in the bosonic case, and this will be corresponding to the Feynman.

Using the same argument, you can show that this is corresponding to the Feynman propagator-- time-ordered propagator of psi. So everything is similar. Yes.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah, so you can just view-- you don't have to think about gamma 0 here. You can just think about the psi-- eta bar just as an independent variable of eta. Yeah.

Yeah, it doesn't matter. You can just treat it as some independent variable. Yes.

AUDIENCE: Are we treating eta as a vector of Grassman variables?

PROFESSOR: No, eta is Grassman. Yeah. Yeah, eta is Grassman.

And in particular, eta depends on space time because we want to be able to take a derivative to get the psi x. And so there are-- yeah, eta has the same-- the eta is the eta 1x eta 4x. Yeah, same with eta bar. Yeah.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Sorry, say it again.

AUDIENCE: Are integrals always complex numbers at the end?

PROFESSOR: No, no, no. The integrals are just some functions of the Grassmann variables. Yeah, just some functions of Grassman variables. Good.

And then from here, from this expression, you can just find all the correlation functions in the free theory. So yeah, let me just quickly write it here. But I don't have-- yeah, I think there is not enough space. Yes.

AUDIENCE: You said that the result of integrating over a Grassmann variable is another Grassmann variable, right? [INAUDIBLE]

PROFESSOR: No, no no. It doesn't have to be. It depends on the situation.

Here, you get a constant. Here, you get the function of a Grassman variable. It just depends on the situation.

AUDIENCE: Oh, OK. So in the path integral, it'll all work out to give a C number at the end.

PROFESSOR: It depends. If you have eta-- so if you don't have eta, then you get a C number. But here, if you have eta, then you get the function of eta. And eta are Grassmann variables. Yeah, mm-hmm.

Good. So now, you can just obtain any correlation functions in Dirac theory. You can just take derivative with respect to this Z. So, say, psi x1 psi xn 0-- and, again, with the previous convention, always use 0 to denote the vacuum of the free theory and then use this omega to denote the vacuum, say, if you have interaction theory.

And so, again, you just get Z0 with eta set to be 0, and then you just do derivatives. Up to a sign-- just delta eta x1 delta eta xn, and then you take derivative on this Z. Yeah, so it depends on whether you do the bar or not the bar.

So, for example, here is the bar. So here, psi-- yeah. And if it's sub psi, then you take derivative with eta bar. If it's psi bar, you take derivative with eta. Anyway, and then you do psi eta equal to eta bar equals 0-- exactly the same as before.

And then again, because these have this kind of structure, you always pair eta bar with eta-- with D-- in here. So when you do that, you just get all possible contractions-- sum of all possible contractions. So each contraction is just a propagator. So if you have x1 bar x2, each contraction is just giving you D0 x1 minus x2. So now I suppress the spinor indices.

So now you have to be careful. So one thing you have to be careful-- you say now all this becomes anticommuting. So when you take the derivative, you have to be careful-- when you do the contraction-- about the order of the psi's.

So, for example, if you look at the 2-point function, if you have psi x1 psi bar x2 and psi bar x3 psi x4-- if you have this, then there are two possible contractions. You can have this contract with that, this contract with that, or this contract with that, and this contract with that. So you have to be careful about the sign.

And if you are careful-- so you find you will get minus D0. So x1 minus x2 and the dx4 minus x3-- so the minus, then, comes from-- you need to exchange these two so that you have the form of psi and the psi bar-- have psi and psi bar. And then you can also have the D0 x1 minus x3 and D0 x4 minus x2.

So in this one, you can just change the order. So here, it should be time-ordered. And so the reason this plus sign is here is because you can shift this x2 by two positions to the right of the psi x4, and then these two will now become neighbors. And then psi 4 and psi 2 will be neighbors. So you just get the positive sign. So you just have to be careful about the sign.

Any questions on this? Other than that, everything is the same as before. So now, you can also just do the interacting theory.

Say, now, the correlation function in the interacting theory divided by omega-- and now, you just do perturbation theory. We just use that, and then you can write it as the correlation functions in free theory. Again, everything is the same, just now you have to be careful now the integral is over the Grassmann variables.

So T-- again, here is T-- time-ordered. Other than that, everything is just the same as before. And again, you can show that a vacuum-- diagrams with vacuum bubbles-- cancel. It will cancel so you don't have to include the vacuum bubble. And then the epsilon prescription is the m goes to m minus i epsilon.

OK, very similar to before. Good. Any questions?

So now, we can just apply, identically, the formalism we developed before for interacting theory of scalar fields to here, except we just have to be careful about some signs, because when you exchange the fermions, you get some signs. And other than those subtleties of signs, everything else will be the same. But those signs sometimes can be annoying.

So we can derive the Feynman-- we can draw the Feynman diagrams as before and write down the Feynman rules, et cetera. So now, let me just write-- yeah, just everything carries over. So let me just emphasize the difference from the scalar case.

So the difference of Feynman rules from the scalar case-- so there's various differences. So first, let's talk about the-- remember, we have different Feynman rules for Green functions and also have ones for scattering amplitudes. So the scattering amplitude is the one which the external legs truncated. And in the Green function, you don't truncate external legs.

So now, let me first say for the Green functions-- so for the propagator-- again, the propagator is just represented by a line. But here, we have a complex field. So as we did for the complex scalar field, now you need to assign an arrow to indicate the flow of the direction of the charge.

So we always assign the arrow to flow from the bar to the unbarred side. So for that one, we will do that. So this will be x2-- will be x1.

Suppose this is beta, this will be alpha. So if I put the alpha beta here, this will be alpha x1 and the beta x2, and then the arrow goes to the-- yeah, so this is a charge arrow. This is not the momentum arrow.

You can also, in additionally, assign a momentum. You can additionally assign a momentum. And the momentum-- yeah, so this is in a coordinate space. Sorry, so this is in the coordinate space. And so this just gives you D0 alpha beta x1 minus x2-- so in the coordinate space.

So in momentum space, we don't label the locations. So in momentum space, we just have alpha and the beta. And now you also have a momentum.

So you can choose the momentum in any direction you want. So, often, for convenience, just as in the scalar case, we often choose a momentum to be in the same arrow in this direction. But in the case which you want to choose a momentum arrow to be different, you can just draw a momentum arrow which can be different.

So the momentum space is the propagator. It is given by what we have already written before. So it's given by minus 1 over i slash minus n plus i epsilon alpha beta. Yes.

AUDIENCE: [INAUDIBLE] I thought of it as a particle traveling through space.

PROFESSOR: Yeah.

AUDIENCE: Like we were contracting two different fields, so why do I still think about it as a fermion moving through space time.

PROFESSOR: Yeah.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah, you can-- no, that aspect is the same as in the complex scalar case.

AUDIENCE: Right, but even then--

PROFESSOR: No different from the complex scalar case. Yeah.

AUDIENCE: But why do we still think of it as the same particle propagating even though it's interacting [INAUDIBLE]

PROFESSOR: Yeah. Yeah, you can consider it as the particle propagating in one direction and then the antiparticle propagating in the opposite direction. You can interpret it either way. Yeah, the importance is the direction of the charge flow.

Good. Yeah, so the momentum we emphasize. So the charge arrow is not arbitrary, but the momentum arrow is arbitrary. So momentum arrow-- you can choose whatever you want.

So also, the rule we do for the external-- so the b-- so if we talk about the external point, we look for the propagator for the correlation functions. Say you have those external points, and some of them will be psi, some of them will be psi bar. Again, we assign the rule as this.

For each external point, if it's a psi-- if the endpoint is given by the psi bar alpha x-- from the rule, the arrow always leaves the direction from the psi bar. Then we should draw it as alpha x. So this is the external point.

And then, similarly, if you have a psi alpha x, this is-- and then the arrow will come in. So in momentum space, if we choose the momentum to be the same direction-- so we just have k, and we take k to be the same arrow. And here, again, it's the same thing.

So this is the coordinate space, this is the momentum space. This is for psi bar, this for psi. Yeah, just follow the same rule. Yes.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah, we'll talk about that. Yeah. Here, I'm talking about the correlation functions. In correlation functions, you don't talk about the particles. You just look at, what are the external points?

And when we talk about scattering amplitudes, then we talk about-- I haven't talked about scattering amplitude yet. Do you have any questions? Other questions?

OK, good. So now the most important-- now you have spinors, and now you have components. Now the propagator is a matrix and each spinor is a vector, so you have to be careful about the indices. So the last thing you have to be careful is that now you have to-- so the spinor indices are contracted following the arrows, and I will explain what this means.

So here, it's a little bit heuristic-- a statement is a bit heuristic-- but I will explain in detail what this means when we talk about scattering amplitudes. Because the scattering ampli-- so this aspect will be similar to the scattering amplitudes that we just explained in one place. So the scattering amplitude-- so the rule follows, again, from this LSZ, which I will not derive-- this Lehmann-Symanzik-Zimmermann reduction rule.

And so you consider-- say you have some initial-- you have some final particles, some initial particles scattered into some final particles. And now we specify the particles not only by their momentum, but also by their polarization. So suppose that the p1 is the momentum of one of the first incoming particles, r1 would be its polarization and et cetera.

And for antiparticle, we put a bar. So for the second particle, it's the antiparticle. Then we put r2 bar.

OK, so that means this is an antiparticle. And similarly, for the final state, we specify it by momentum and the polarization. And if we do a bar, again, we mean the antiparticle.

The scattering amplitude is where we want to calculate things like this with some initial momentum and the-- so to specify by the initial and the final momenta and the polarizations. And then the bar is for antiparticle. So between the scattering amplitude and the Green function, the only difference is how you choose the external lines. All the propagators-- these are the same.

So now, remember in the scalar case, we just remove all the external lines. But here, we have to be a little bit careful because now they have the polarization. And now we should have things to specify the polarization of each particle. So now it's reasonable to expect, and the polarization of each particle is specified by those u and v functions we derived earlier.

So we truncate all the propagators for the external lines, but we have to assign polarization vectors. So need to-- assign polarization vectors for each initial and final particle-- particle or antiparticle. So let me just state the rule, and then I will motivate the rule.

So for the initial state, if you consider the initial state-- then if you have a particle-- so suppose this particle is polarization-- r1. And then we draw a line like this with the arrow like that, and then assign the polarization vector, assign u r1. So suppose that these have momentum p1 and p1.

And for antiparticle, say, for example, r2 bar-- say we just reverse the direction of the arrow. And then the polarization vector-- I use v bar r2 and p2. I always draw the momentum to be the same as the charge arrow.

So this is for the initial state, and the final state is the following. So we just first write down the rule, then I will motivate it. So the final state-- if it's a particle, then the particle will come out.

And this is s1 and, say, suppose this is k1. And then it's given by u bar s1 k1. So if it's an antiparticle, and then it's an arrow which is going out-- suppose this is s2 bar, then this is given by v2 vs2 k2.

So when you do the scattering amplitude, you forget about the external propagator, but you attach to each external line a polarization vector. OK, polarization vector-- just specify u and v according to this rule. So now, let's see why this is the rule. So remember-- so let's use this as an example.

So this is the initial state, then that means this is a ket. So if you remember that psi has the structure-- the au plus b dagger v-- then psi bar has the structure a dagger u bar plus b, say, v bar. So remember this structure.

So for the initial state, let's try to motivate this rule. So initial state, you have a p1 r1. So suppose you have a particle like this. So this is obtained by-- so don't worry about the proportional factor.

So this is given by p1, then r1 dagger acting on 0. And then this you can obtain-- you see the psi is proportional-- so in order to have a dagger, you have to have psi bar. So this is given by-- it can be obtained from psi bar by multiplying psi bar, say, by something like this.

So exponential minus i-- yeah, so you can verify this, but, heuristically, you will understand. Psi bar x gamma zero u r1 p1-- acting on 0. So because psi bar has lots of other things, you need to multiply this by this polarization vector so that it will extract the a p1 r1 piece. And so this is the polarization vector you need to include here.

And the reason this is the arrow which is going out is because this is psi bar. So remember, the things always come out from the psi bar-- from this rule, to come out of this from psi bar. So similarly, you can understand the other rules. Any questions? Yes.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah, because you have a spin. Yeah, a spin has polarization. This is to characterize the polarization of the spin. Yeah.

AUDIENCE: I'm just trying to think back-- is there a symmetry that causes us to need to specify this polarization?

PROFESSOR: Sorry?

AUDIENCE: So this is always the case for fermions.

PROFESSOR: Yeah. Right, right. Yeah, this is always-- yeah.

AUDIENCE: When I'm thinking back to the scalar piece [INAUDIBLE]

PROFESSOR: Yeah, because there's no polarization. Yeah, for scalar, all this just becomes one because there's no polarization. Other questions?

OK, so these are just rules you can understand. And then this antiparticle initial state will be created by psi, and then you need to multiply by psi-- by v bar on this term to extract this b dagger term. Because the v v bar, they're contracted.

And then you can extract this v plus. Yeah, so that's where this will come. Yes.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah, in the real experiment, it's often-- it's not easy to observe the polarized. You have to have very special-- yeah. In general, you observe unpolarized. Other questions? Good.

So again, now I can specify more precisely this spinor indices rule for the scattering amplitude case. So now you see the initial state, so now you have this u bar and you have u. So the rule for the spinor indices contraction are the following.

So the spinor indices are contracted by starting at the one end, so at one end of a fermionic line-- this external factor, either u bar or v bar. So you start from-- and then you go back, and then you go along the complete line following the arrow backwards. So right now, it's a little bit abstract.

Now let me just explain this rule using an example. It would be very clear. Now, we'll explain this rule using an example.

So now let's consider the following example. So a very important example in the development of particle physics is this called Yukawa theory, which Yukawa proposed, I think, around the 1930s, something-- and which he got a Nobel Prize for it. So let's consider you have a scalar field. So let's just say you have the Klein-Gordon-- Lagrangian density for scalar fields, and then you have a Dirac Lagrangian for psi. And then, supposed they intact, a term like this-- minus g phi psi bar psi.

So this clearly is Lorentz invariance. And so, essentially, now you can just draw the propagator. So the propagators-- say, let's denote this to be the phi, and then these have the standard minus i, say, k squared plus m squared minus i epsilon. So if this has momentum k-- so we also have fermionic lines. I will draw it using the real line so this can have the form ik slash plus m minus m plus i epsilon.

And then we have interaction vertices-- so minus ig. So because here, I have a psi and a psi bar-- one of the lines coming out, one of the lines coming in. So now, let's just consider a scattering process. When Yukawa considered this, and the psi corresponding to say, the proton-- and then the phi corresponding to pion. And he used this theory to explain the nuclear force between the proton.

So now let's just consider the proc-- yeah, but you can also consider, say, psi is an electron or phi is a Higgs field, et cetera. So now let's just consider the process of this p. So let's denote the psi particle by p, go to p. So you have two initial states. Particles in final state are also particles.

And now you have the one obvious diagram you can draw, because now the internal-- so the incoming line should be-- because the particles should go in, and then the final state, the particle should come out. So one possible diagram is like this. So suppose this is the p1. Polarization s1-- suppose this is p2 s2, and then this is k1.

So let's call it p1 prime s1 prime and p2 prime s2 prime. So this is one of the diagrams which can do this, but we can also have this internal line connect with that line and this line connect that line. So you can also have a structure like this. So we have p1 s1 and p2 s2, then p1 prime s1 prime p2 prime s2 prime. OK, so these are the two possible diagrams, at lowest order.

So now let's write down the expression corresponding to these two diagrams. So the first thing we can write down-- so each thing should give you a number in the end. So that's why-- but remember, all these are vectors. All these are spinors. So in the end, all the spinor indices should be contracted with each other.

So that's why when we do this rule, you always start with u bar and the v bar, because these are the row vectors. In the end, you want to contract with column vectors. So that's why you always start with u bar and the v bar.

So u bar and the v bar-- if you look at this rule here-- so the u bar either corresponding to the final particles, v bar corresponding to initial antiparticles. In both cases, the arrow is always going toward the point. If you start from here, you have to go backwards.

So now let's look at this example. So we have two fermionic lines here. So one is this one and one is this one. OK, we have two fermionic lines.

So for this expanded line, we should start with the bar. So this is the outgoing particle, so that's corresponding to u bar here, s1 prime, and here corresponding to u bar s2 prime. And then this is corresponding to-- this is the initial particle, then this is corresponding to u s1, and this is corresponding to u s2. And so we just go backwards.

So now we can just write it down very easily. So we have minus ig squared corresponding to two vertices, and for the first fermionic line, we have-- for the first diagram, we have u s1 prime bar p1 prime and then u s1 p1. And so these combine into a sca-- number.

And now we have the propagator. A propagator is just a number minus i p1 prime minus p2 squared plus m squared minus i epsilon. So in principle, they can have a different mass. OK, sorry. For example, here is m tilde.

And then we multiply the other fermionic line. So this will be u s2 prime bar p2 prime and then u s2 p2. And now, for this diagram, we just follow the same rule. But now, this one is contracted with that, and this one is contracted with that-- so except we still have minus ig squared.

So now this u s1 prime p1 prime is contracted with u s2 p2. OK, they are connected. Now they become a connected line. And, again, you have these indices, and now you work out the momentum to become p2 prime minus p1 squared plus m squared minus i epsilon. And then you have an s2 prime p2 prime, and now you have u s1 p1.

Except one final thing, there's a relative minus sign between them. The reason there's a relative minus sign between them is because between these two diagrams, it's equivalent. I exchanged the order of the two external legs.

So remember, if you want to exchange the order of the fermions, you have to have a minus sign. And then you have to have a minus sign between the two. Yes.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Oh. Yeah, of course. But we never write them down explicitly. So always assume momentum conservation. Yeah.

So for this diagram, the momentum-- if I draw momentum here, it will be p prime prime minus p1. So this momentum will be p2 prime minus p1. Yes.

AUDIENCE: So it's probably [INAUDIBLE]

PROFESSOR: Minus p1-- that's right. Yeah, that's right. Good?

So just like that, you just follow this rule. You just start with the line. Always start with u bar and v bar, and you just follow the arrow backwards.

And keep track all the gamma matrices and the spinor indices, et cetera. And then, eventually, you will just get a number. You will multiply with a column vector, and then you are done.

So here, we are considering-- here, the vertex is very simple. It's just a number. And later, we will consider a more complicated vertex. And actually, the vertex can contain matrices. And so that will be a little bit more complicated.

Also, this is a simple-- here is a simple example. We don't have a fermionic propagator. We only have a bosonic propagator here. Yes.

AUDIENCE: How do you know which one's which?

PROFESSOR: Yeah, it doesn't matter because the overall signs don't matter. Because in the end, we will square it. Yeah, so when you calculate the cross section, you have to take the-- yeah, in order to calculate the probability, you have to take the amplitude square. So overall, signs we don't normally bother to track.

Yeah, you only need to worry about relative signs. Other questions? Yes.

AUDIENCE: Where did the use of Grassmann variables [INAUDIBLE]

PROFESSOR: Yeah, the sign-- so the relative sign between them. That's the only place. For this example, that's the only place. Yeah, so that's the minus sign. And, of course, the spinor later is reflected that you always have to keep track of the spinor indices.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah, indeed. And then you just miss this minus sign, essentially.

AUDIENCE: Could you have gotten the minus sign if you just used the operators, right?

PROFESSOR: Yeah, but you have to know that you have to exchange them to get the minus sign. You have to have that. Yeah, if you don't have that-- I think when Yukawa proposed it, he actually didn't know all this detail. He just estimated.

And then he predicted the-- yeah, he wanted to explain the nuclear force between the proton, and then he said, oh, maybe the two protons will exchange all between proton or neutron. Or maybe they exchange a scalar particle. And he just postulated the exchange of a scalar particle.

And then from the strength of the nuclear interaction, he estimated the mass of the particle-- the scalar particle-- and then they found it. Yeah. [LAUGHS] Yeah, they found such a scalar particle which was the pion. And that's how they first discovered the pion. Yeah.

OK, good. Let's stop here today.