Quantum Field Theory I (8.323) Spring 2023
Assignment 2

Feb. 14, 2023

- Please remember to put your name at the top of your paper.

Readings
- Peskin & Schroeder Chap. 2
- Weinberg vol 1 Chap. 1

Notes:

1. Conventions on Fourier transform and the Dirac delta function
   - Fourier transform of $\phi(\vec{x}, t)$ is defined as
     \[ \tilde{\phi}(\vec{k}, \omega) = \int dtd^3\vec{x} e^{i\omega t - i\vec{k} \cdot \vec{x}} \phi(\vec{x}, t) \]  
     with the inverse transform given by
     \[ \phi(\vec{x}, t) = \int d\omega \frac{d^3 \vec{k}}{(2\pi)^3} e^{-i\omega t + i\vec{k} \cdot \vec{x}} \tilde{\phi}(\vec{k}, \omega). \]  
     We will often suppress the tilde on $\tilde{\phi}(\vec{k}, \omega)$ and simply write it as $\phi(\vec{k}, \omega)$, distinguishing it from $\phi(\vec{x}, t)$ by their arguments.
   - Note
     \[ \int_{-\infty}^{\infty} dx \, e^{ikx} = 2\pi \delta(k), \]  
     and it higher dimensional generalizations
     \[ \int d^3\vec{x} \, e^{i\vec{k} \cdot \vec{x}} = (2\pi)^3 \delta^{(3)}(\vec{k}). \]  

2. Lorentz transformations
• A Lorentz transformation acts on $x^\mu$ and $p^\mu$ as

$$x^\mu \rightarrow x'^\mu = \Lambda^\mu_\nu x^\nu, \quad p^\mu \rightarrow p'^\mu = \Lambda^\mu_\nu p^\nu$$

(5)

where the matrix $\Lambda^\mu_\nu$ satisfies the relation

$$\Lambda^\mu_\rho \Lambda^\rho_\lambda \eta^{\rho\lambda} = \eta^{\mu\nu}$$

(6)

or in a matrix notation

$$\Lambda \eta \Lambda^t = \eta$$

(7)

where the superscript $t$ denotes transpose. We can raise and lower the indices of $\Lambda$ by $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$, and equation (7) can also be written as

$$\Lambda_\mu^\rho \Lambda_\nu^\lambda \eta_{\rho\lambda} = \eta_{\mu\nu} .$$

(8)

• Under a Lorentz transformation (5), a scalar field transforms as

$$\phi(x) \rightarrow \phi'(x') = \phi(x) ;$$

(9)

a vector field transforms as

$$A_\mu(x) \rightarrow A'_\mu(x') = \Lambda^\mu_\nu A_\nu(x) ;$$

(10)

a second rank tensor field transforms as

$$T_{\mu\nu}(x) \rightarrow T'_{\mu\nu}(x') = \Lambda^\lambda_\mu \Lambda^\rho_\nu T_{\lambda\rho}(x)$$

(11)

and so on.

• Infinitesimal Lorentz transformations take the form

$$\Lambda_\mu^\nu = \delta_\mu^\nu + \omega_\mu^\nu$$

(12)

where

$$\omega_{\mu\nu} = -\omega_{\nu\mu}, \quad \omega_\mu^\nu = \eta^{\mu\lambda} \omega_{\nu\lambda}$$

(13)

are infinitesimal numbers.

3. All single-particle states used below follow relativistic normalization, i.e.

$$|k\rangle = \sqrt{2\omega_k} a^\dagger_k |0\rangle .$$

(14)

Problem Set 2
1. **Problem with relativistic quantum mechanics (20 points)**

The Schrödinger equation for a free non-relativistic particle is

\[ i\partial_t \psi(\vec{x}, t) = -\frac{1}{2m} \nabla^2 \psi(\vec{x}, t) \, . \]  

(15)

The generalization of the above equation to a free relativistic particle is the so-called Klein-Gordon equation

\[ \partial_t^2 \psi(\vec{x}, t) - \nabla^2 \psi(\vec{x}, t) + m^2 \psi(\vec{x}, t) = 0 \, . \]  

(16)

We emphasize that in both (15) and (16), \( \psi(\vec{x}, t) \) is interpreted as a wave function for dynamical variable \( \vec{x}(t) \) rather than a dynamical field.

(a) As a reminder, derive from (15) the continuity equation for the probability

\[ \partial_t \rho + \nabla \cdot \vec{J} = 0, \]  

(17)

where

\[ \rho = |\psi|^2, \quad \vec{J} = -\frac{i}{2m} (\psi^* \nabla \psi - \psi \nabla \psi^*) \, . \]  

(18)

(b) Suppose \( \psi(\vec{x}, t) \) has the plane wave form, i.e.

\[ \psi(\vec{x}, t) \propto e^{i\vec{k} \cdot \vec{x}} \]  

(19)

for some real vector \( \vec{k} \), find the solutions to (16).

(c) Show that the Klein-Gordon equation also leads to a continuity equation (17) with now \( \rho \) and \( \vec{J} \) given by

\[ \rho = \frac{i}{2m} (\psi^* \partial_t \psi - \psi \partial_t \psi^*) , \quad \vec{J} = -\frac{i}{2m} (\psi^* \nabla \psi - \psi \nabla \psi^*) \, . \]  

(20)

(d) Argue that \( \rho \) in (20) cannot be interpreted as probability density.

2. **Commutation relations of annihilation and creation operators (20 points)**

For the real scalar field theory discussed in lecture, i.e.

\[ \mathcal{L} = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \]  

(21)

we showed that the time evolution of quantum operator \( \phi(\vec{x}, t) \) is given by

\[ \phi(\vec{x}, t) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left( a_k \phi_k(\vec{x}, t) + a_k^* \phi^*_k(\vec{x}, t) \right) \]  

(22)
where

\[
\omega_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}, \quad u_{\vec{k}}(\vec{x}, t) = e^{-i\omega_{\vec{k}}t + i\vec{k} \cdot \vec{x}}.
\]  

(23)

We use \(\pi(\vec{x}, t)\) to denote the momentum density conjugate to \(\phi\). The canonical commutation relations among \(\phi\) and \(\pi\) are

\[
[\phi(\vec{x}, t), \phi(\vec{x}', t)] = 0 = [\pi(\vec{x}, t), \pi(\vec{x}', t)], \quad [\phi(\vec{x}, t), \pi(\vec{x}', t)] = i\delta^{(3)}(\vec{x} - \vec{x}').
\]  

(24)

(a) Show that it is enough to impose (24) at \(t = 0\). In other words, once we impose them at \(t = 0\), then the relations at general \(t\) are automatically satisfied.

Note: This statement in fact applies not only to \(V(\phi) = \frac{1}{2}m^2\phi^2\), but any potential \(V(\phi)\).

(b) Express \(a_{\vec{k}}\) and \(a_{\vec{k}}^\dagger\) in terms of \(\phi(\vec{k})\) and \(\pi(\vec{k})\), where \(\phi(\vec{k})\) and \(\pi(\vec{k})\) are Fourier transforms of \(\phi(\vec{x}, t = 0)\) and \(\pi(\vec{x}, t = 0)\), i.e.

\[
\phi(\vec{k}) = \int d^3 x e^{-i\vec{k} \cdot \vec{x}} \phi(\vec{x}, t = 0)
\]  

(25)

and similarly for \(\pi\).

(c) Using the expressions you derived in part (b) to deduce the commutations relations

\[
[a_{\vec{k}}, a_{\vec{k}}], \quad [a_{\vec{k}}^\dagger, a_{\vec{k}}^\dagger], \quad [a_{\vec{k}}, a_{\vec{k}}^\dagger]
\]  

(26)

from the commutation relations (24) at \(t = 0\).

3. Expressing Noether charges in terms of creation and annihilation operators (20 points)

In pset 1 you obtained the conserved charges associated with spacetime translational symmetries for a complex scalar field theory. The results there can be easily converted to the corresponding expressions for a real scalar field theory (21).

(a) Express the Hamiltonian \(H\) of (21) in terms of \(a_{\vec{k}}\) and \(a_{\vec{k}}^\dagger\).

(b) Express the conserved charges \(P^i\) for spatial translations for (21) in terms of \(a_{\vec{k}}\) and \(a_{\vec{k}}^\dagger\).

(c) Starting with

\[
\phi(0, 0) = \int \frac{d^3 k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} \left( a_{\vec{k}} + a_{\vec{k}}^\dagger \right)
\]  

(27)

show that under the action of translation operators

\[
\phi(\vec{x}, t) = e^{iHt - iP^ix^i} \phi(0, 0) e^{-iHt + iP^ix^i}.
\]  

(28)
Note: This problem becomes trivial if you recall the following formula for a harmonic oscillator
\[ e^{i\alpha N} a e^{-i\alpha N} = e^{-i\alpha a}, \quad N = a^\dagger a \] (29)
and \( \alpha \) is a constant.

4. Noether charges for Lorentz symmetries of the real scalar field theory (20 points + 10 bonus points)

In this problem we work out the conserved current corresponding to Lorentz symmetries of (21).

(a) Consider an infinitesimal Lorentz transformation (12)–(13). Show that (12) satisfies (6) to first order in \( \omega_{\mu\nu} \), so does give a Lorentz transformation.

(b) Write down how \( \phi \) transforms under an infinitesimal Lorentz transformation (see (9)) and show that the conserved Noether current for this transformation can be written as
\[ J_{\mu\lambda\nu} = x^{\lambda} T_{\mu\nu} - x^{\nu} T_{\mu\lambda} \] (30)
where \( T_{\mu\nu} \) is the conserved energy-momentum tensor which we have already derived in pset 1.

Note: this part does not involve complicated calculations. If you find yourself in a massive calculation, pause, and try to find a simpler approach.

(c) Use the conservation of the energy-momentum tensor to verify that the current (30) is indeed conserved, i.e.
\[ \partial_{\mu} J_{\mu\lambda\nu} = 0. \] (31)

This problem is complete if you finish the above parts. The part below is an instructive exercise, but is calculation heavy. It is given as a bonus problem (10 extra points) for those of you who would like to have more fun.

(d) Consider the conserved charges associated with \( J_{\mu\lambda\nu} \)
\[ M_{\lambda\nu} = \int d^3 x J^{0\lambda\nu} \] (32)
Express the conserved charges \( M_{\mu\nu} \) for Lorentz symmetries for (21) in terms of \( a_k^\dagger \) and \( a_k \).