Quantum Field Theory I (8.323) Spring 2023
Assignment 4

Feb. 28, 2023

- Please remember to put **your name** at the top of your paper.

Readings

- Peskin & Schroeder Chap. 4.1 – 4.4
- Peskin & Schroeder Chap. 9.1–9.2

Notes:

1. Here we give the definitions of various functions for a complex scalar field ϕ:

$$\text{Wightman function } \; G_+(x,x') = \langle 0|\phi(x)\phi^\dagger(x')|0\rangle \tag{1}$$

$$\text{retarded function } \; G_R(x,x') = \theta(t-t')\Delta(x,x') \tag{2}$$

$$\text{advanced function } \; G_A(x,x') = -\theta(t'-t)\Delta(x,x') \tag{3}$$

where

$$\Delta(x,x') = \langle 0|[\phi(x),\phi^\dagger(x')]|0\rangle \tag{4}$$

and the Feynman function

$$\begin{aligned} G_F(x,x') &= \langle 0|T\phi(x)\phi^\dagger(x')|0\rangle \\ &= \theta(t-t')\langle 0|\phi(x)\phi^\dagger(x')|0\rangle + \theta(t'-t)\langle 0|\phi^\dagger(x')\phi(x)|0\rangle \, . \end{aligned} \tag{5}$$

Note that these definitions may differ from various books by an overall factor i or minus sign.

2. The unitary quantum operator generating a spacetime translation y^μ is written as

$$U_y = e^{-iP^\mu y_\mu} = e^{iHy^0 - iP^i y^i}, \qquad P^\mu = (H, P^i) \, . \tag{6}$$

It should be understood that the vacuum energy is already subtracted from H. U_y acts on creation and annihilation operators as

$$U_y a_{\vec{k}} U_y^\dagger = e^{-i\omega_{\vec{k}} y^0 + ik^i y_i} a_{\vec{k}} = e^{ik\cdot y} a_{\vec{k}}, \qquad U_y a_{\vec{k}}^\dagger U_y^\dagger = e^{-ik\cdot y} a_{\vec{k}}^\dagger \, . \tag{7}$$

3. The quantum operator generating a Lorentz transformation is

$$U_\Lambda = e^{\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}} \tag{8}$$

where $\omega_{\mu\nu} = -\omega_{\nu\mu}$ are finite constants. It acts on creation and annihilation operators as

$$U_\Lambda a_{\vec{k}} U_\Lambda^\dagger \quad = \quad = \sqrt{\frac{\omega_{\Lambda\vec{k}}}{\omega_{\vec{k}}}} a_{\Lambda\vec{k}} \tag{9}$$

$$U_\Lambda a_{\vec{k}}^\dagger U_\Lambda^\dagger \quad = \quad = \sqrt{\frac{\omega_{\Lambda\vec{k}}}{\omega_{\vec{k}}}} a_{\Lambda\vec{k}}^\dagger \ . \tag{10}$$

Problem Set 4

1. **Properties of Wightman and Feynman functions (20 points)**

   (a) For a free scalar field theory, using (9) and (10), show that

   $$U_\Lambda \phi(x) U_\Lambda^\dagger = \phi(\Lambda x) \ . \tag{11}$$

   (b) Now in all the subsequent parts, let us consider a general interacting theory, where the mode expansion given in lecture for ϕ does *not* apply. But as far as the system is translational and Lorentz invariant, both (11) and (below U_y is given by (6))

   $$U_y \phi(x) U_y^\dagger = \phi(x+y) \tag{12}$$

   are valid[1]. Using (12) to prove that for Wightman function introduced in (1)

   $$G_+(x, x') = G_+(x - x') \ . \tag{13}$$

   You need not to show it for other functions, but should realize the same is true for all two-point functions defined at the beginning of this pset.

   (c) Using (11) and the result of part (b) show that one can write $G_+(x, x')$ as

   $$G_+(x, x') = \theta(t - t')G((x - x')^2) + \theta(t' - t)G^*((x - x')^2) \tag{14}$$

   where $G(y)$ is some function which satisfies

   $$G(y) = G^*(y), \qquad \text{for } y > 0 \ . \tag{15}$$

[1] One can prove this by using the expressions of H, P^i and $M^{\mu\nu}$ in terms of ϕ and π_ϕ and canonical commutation relations between ϕ and π_ϕ.

(d) Now take ϕ to be real and show that for Feynman function G_F

$$G_F(x, x') = G((x - x')^2) \tag{16}$$

where function G in the above equation is the same as that in (14).

2. **Particle production by an external source (60 points)**

This problem is developed from Peskin and Schroeder's example on p. 32 and prob. 4.1. It is the simplest example in which the S-matrix can be calculated exactly.

Consider a free scalar field theory with an external "source" $J(x)$, whose Lagrangian density can be written as

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + J(x)\phi = \mathcal{L}_0 + J(x)\phi \tag{17}$$

where $\mathcal{L}_0$ is the Lagrangian density for a free scalar and $J(x)$ is a fixed function. We assume that it has the the following properties

$$J(t, \vec{x}) \to 0, \qquad t \to \pm\infty, \quad |\vec{x}| \to \infty \tag{18}$$

and its Fourier transform

$$J(p) = \int d^4x \, e^{-ip\cdot x} J(x) \tag{19}$$

is analytic in the complex ω plane.

Since $J(x)$ depends on time, the system does not have time translation symmetry. In particular, the vacuum at past infinity $|0, -\infty\rangle$ will be *different* from the one at future infinity $|0, +\infty\rangle$. Suppose we start with the vacuum state $|0, -\infty\rangle$ at $t = -\infty$, in the Heisenberg picture, the system remains in the same state $|0, -\infty\rangle$ at all times. At $t = +\infty$, the system is then *not* in the ground state (as $|0, -\infty\rangle \neq |0, +\infty\rangle$), and contains particle excitations. In other words, turning on a source $J(x)$ has produced particles. Below we will find the relation between $|0, -\infty\rangle$ and $|0, +\infty\rangle$, and calculate the probability for producing particles.

At $t = -\infty$, since $J = 0$, we have a free theory, and ϕ can be written as

$$\phi(x) = \phi_{in}(x) \equiv \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left[a_{in}(\vec{k}) e^{ik\cdot x} + a^\dagger_{in}(\vec{k}) e^{-ik\cdot x} \right], \qquad t \to -\infty \tag{20}$$

where

$$[a_{in}(\vec{k}), a^\dagger_{in}(\vec{k}')] = (2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') \ . \tag{21}$$

We can then define the past vacuum as

$$a_{in}(\vec{k})|0, -\infty\rangle = 0, \qquad \langle 0, -\infty|0, -\infty\rangle = 1 \tag{22}$$

and particles (in the past infinity) can be defined by acting $a_{in}^\dagger$ on this vacuum. For example an n-particle state in the past infinity can be written as

$$|\vec{k}_1, \cdots \vec{k}_n, -\infty\rangle = \sqrt{2\omega_{\vec{k}_1}} \cdots \sqrt{2\omega_{\vec{k}_n}} a_{in}^\dagger(\vec{k}_1) \cdots a_{in}^\dagger(\vec{k}_n)|0, -\infty\rangle \tag{23}$$

with normalization for a single-particle state

$$\left\langle \vec{k}, -\infty | \vec{k}', -\infty \right\rangle = 2\omega_{\vec{k}}(2\pi)^3 \delta^{(3)}(\vec{k} - \vec{k}') \ . \tag{24}$$

Similarly, at $t = +\infty$, since $J = 0$, we again have a free theory, and ϕ can be written as

$$\phi(x) = \phi_{out} \equiv \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left[a_{out}(\vec{k}) e^{ik\cdot x} + a_{out}^\dagger(\vec{k}) e^{-ik\cdot x} \right], \qquad t \to +\infty \ . \tag{25}$$

with the future vacuum defined as

$$a_{out}(\vec{k})|0, +\infty\rangle = 0, \qquad \langle 0, +\infty | 0, +\infty\rangle = 1 \tag{26}$$

and an n-particle state written as

$$|\vec{k}_1, \cdots \vec{k}_n, +\infty\rangle = \sqrt{2\omega_{\vec{k}_1}} \cdots \sqrt{2\omega_{\vec{k}_n}} a_{out}^\dagger(\vec{k}_1) \cdots a_{out}^\dagger(\vec{k}_n)|0, +\infty\rangle \ . \tag{27}$$

Due to the presence of the external source $J(x)$, the past and future annihilation operators a_{in} and a_{out} are different. $\phi(t, \vec{x})$ with $-\infty < t < +\infty$ interpolates between (20) and (25).

We consider that the system starts in the vacuum, i.e.

$$|\Psi\rangle = |0, -\infty\rangle \ . \tag{28}$$

(a) For general t, solve the classical equation of motion for (17) and show that the solution can be written in a form

$$\phi(x) = \phi_0(x) + i \int d^4x \, G(x - x') J(x') \tag{29}$$

where $\phi_0(x)$ is a solution of the homogeneous equation

$$(-\partial^2 + m^2)\phi_0(x) = 0 \tag{30}$$

and $G(x - x')$ is a Green function satisfying

$$(-\partial^2 + m^2) G(x - x') = -i\delta^{(4)}(x - x') \ . \tag{31}$$

We have discussed various types of Green functions, the retarded, advanced, and Feynman Green functions. Which one we should use here?

(b) Now consider the quantum theory and promote ϕ to a quantum operator, with (29) now an operator equation. Show that ϕ_0 in (29) should be given by ϕ_{in} introduced in (20).

(c) Evaluate (29) at $t = +\infty$ to find the relation between $a_{out}(\vec{k})$ and $a_{in}(\vec{k})$.

(d) Using (c) show that the expectation value λ for the total number of particles produced is given by

$$\lambda = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} |J(k)|^2 \ . \tag{32}$$

(e) Show that we can write

$$a_{out}(\vec{k}) = S^\dagger a_{in}(\vec{k}) S, \tag{33}$$

with S a unitary operator given by

$$S \equiv e^{iB}, \qquad B = \int d^4x \, J(x) \phi_{in}(x) \ . \tag{34}$$

(f) Use (33) to show that

$$S|0, +\infty\rangle = |0, -\infty\rangle \tag{35}$$

and

$$S|\vec{k}_1, \cdots \vec{k}_n, +\infty\rangle = |\vec{k}_1, \cdots \vec{k}_n, -\infty\rangle \ . \tag{36}$$

Note: The results of this part indicate that S is in fact the S-matrix operator of the system, i.e. for any free theory state $|\alpha\rangle$ and $|\beta\rangle$,

$$S_{\beta\alpha} \equiv \langle \beta, +\infty | \alpha, -\infty \rangle = \langle \beta, -\infty | S | \alpha, -\infty \rangle \ . \tag{37}$$

(g) In the following parts we will compute the probability of producing n particles. Before doing that, in this part we develop a technical tool to make that task easier. Show that we can write S as

$$S = e^{iB} = e^F e^G e^{-\frac{1}{2}\lambda} \tag{38}$$

with

$$F = i \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} J(k) a_{in}^\dagger(k), \quad G = i \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} J(-k) a_{in}(k) \tag{39}$$

and λ was introduced in (32).

(h) Use (35) and (38) to find the vacuum to vacuum probability , i.e.

$$P_0 = |\langle 0, +\infty | 0, -\infty \rangle|^2 \ . \tag{40}$$

P_0 is the probability of no particle production.

(i) Use (36) and (38) to show that

$$\left\langle \vec{k}_1, \cdots, \vec{k}_n, +\infty | 0, -\infty \right\rangle = i^n J(k_1) \cdots J(k_n) e^{-\frac{1}{2}\lambda} \ . \tag{41}$$

(j) The probability dP of finding exactly n particles with one particle in the range $d^3\vec{k}_1$ around $\vec{k}_1$, one particle in the range $d^3\vec{k}_2$ around $\vec{k}_2$, $\cdots$, one particle in the range $d^3\vec{k}_n$ around $\vec{k}_n$ can be shown to be given by

$$dP = \left| \left\langle \vec{k}_1, \cdots, \vec{k}_n, +\infty | 0, -\infty \right\rangle \right|^2 \prod_{i=1}^{n} \left(\frac{d^3\vec{k}_i}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}_i}} \right) \ . \tag{42}$$

Use (42) to show that the probability of finding exactly n particles is

$$P_n = e^{-\lambda} \frac{\lambda^n}{n!} \ . \tag{43}$$

This is Poisson distribution with average particle number λ.

[*Note:* I am not asking you to derive (42). The equation of course looks quite intuitive and many of you may be able to guess it. We will discuss its precise derivation a bit later.]

Remarks on problem 2:

1. While the setup of this problem is very simple, the notion that in a time-dependent situation the past and future vacua are inequivalent (and the resulting particle production) is very general and plays very important roles in cosmology and black holes. In particular, the famous Hawking radiation from black holes is also a consequence of this.

2. We notice that only the on-shell part of $J(k)$ (i.e. with $k^0 = \omega_k$) contributes to the particle production. This is just field theory analog of the resonance effect for a forced harmonic oscillator.

3. Suppose we turn on the source J very slowly and very slowly turn it off, from the adiabatic theorem of quantum mechanics, we expect the system to remain in the vacuum throughout, i.e. there is no particle production. Indeed to see this in our discussion, in the adiabatic limit $J(\omega, \vec{k})$ will be nonzero only for ω close to zero and there is no on-shell component.

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