Quantum Field Theory I (8.323) Spring 2023
Assignment 8

Apr. 4, 2023

- Please remember to put **your name** at the top of your paper.

Readings

- Peskin & Schroeder Chap. 3

Notes: conventions and some useful formulae

1. Conventions of γ matrices:

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{1}$$

and

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0 \ . \tag{2}$$

2. The Dirac equation has the form

$$(\gamma^\mu\partial_\mu - m)\psi = 0 \tag{3}$$

and the action is given by

$$S = -i\int d^4x\, \bar{\psi}(\partial\!\!\!/ - m)\psi \ . \tag{4}$$

3. A spinor ψ transforms under a Lorentz transformation Λ as

$$\psi'_\alpha(x') = S_\alpha{}^\beta(\Lambda)\psi_\beta(x), \qquad x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{5}$$

with

$$\Lambda^\mu{}_\nu = \left(e^{-\frac{i}{2}\omega_{\lambda\rho}\mathcal{J}^{\lambda\rho}}\right)^\mu{}_\nu, \qquad S(\Lambda) = e^{-\frac{i}{2}\omega_{\lambda\rho}\Sigma^{\lambda\rho}}, \tag{6}$$

and

$$\left(\mathcal{J}^{\lambda\rho}\right)^\mu{}_\nu = i\left(\eta^{\lambda\mu}\delta^\rho{}_\nu - \eta^{\rho\mu}\delta^\lambda{}_\nu\right), \qquad \Sigma^{\lambda\rho} = \frac{i}{4}[\gamma^\lambda, \gamma^\rho] \ . \tag{7}$$

4. Note the relations

$$[\Sigma^{\lambda\rho}, \gamma^\mu] = -\left(\mathcal{J}^{\lambda\rho}\right)^\mu{}_\nu\gamma^\nu, \tag{8}$$

$$S(\Lambda)\gamma^\mu S^{-1}(\Lambda) = (\Lambda^{-1})^\mu{}_\nu\gamma^\nu, \tag{9}$$

$$S^\dagger = -\gamma^0 S^{-1}\gamma^0 \tag{10}$$

5. $u_s(\vec{k})e^{ik\cdot x}$ and $v_s(\vec{k})e^{-ik\cdot x}$, $s = 1, 2$ denote respectively a basis of positive and negative energy solutions to the Dirac equation, with $k^2 = -m^2$.

6. We normalize $u_s(\vec{k})$ and $v_s(\vec{k})$ as

$$\bar{u}_r(\vec{k})u_s(\vec{k}) = 2mi\delta_{rs}, \qquad \bar{v}_r(\vec{k})v_s(\vec{k}) = -2mi\delta_{rs} \ . \tag{11}$$

   $u_s(\vec{k})$ and $v_s(\vec{k})$ are orthogonal

$$\bar{u}_r(\vec{k})v_s(\vec{k}) = 0, \qquad \bar{v}_r(\vec{k})u_s(\vec{k}) = 0 \ . \tag{12}$$

7. With normalization (11), we have

$$u_r^\dagger(\vec{k})u_s(\vec{k}) = 2E\delta_{rs}, \qquad v_r^\dagger(\vec{k})v_s(\vec{k}) = 2E\delta_{rs} \ , \tag{13}$$

   and the orthogonal relations (12) can also be written as

$$u_r^\dagger(\vec{k})v_s(-\vec{k}) = 0, \qquad v_r^\dagger(\vec{k})u_s(-\vec{k}) = 0 \ . \tag{14}$$

   These relations are valid for any choices of basis and any representation of gamma matrices once the normalizations are fixed as in (11).

8. With normalization (11), one can also show that

$$\Lambda_+(\vec{k}) = \sum_{s=1,2} u_s(\vec{k}) \otimes \bar{u}_s(\vec{k}) = i(i\not{k} + m), \tag{15}$$

$$\Lambda_-(\vec{k}) = \sum_{s=1,2} v_s(\vec{k}) \otimes \bar{v}_s(\vec{k}) = -i(-i\not{k} + m) \ . \tag{16}$$

9. An operator solution $\psi(x)$ to the Dirac equation can be expanded as

$$\psi(x) = \int \frac{d^3\vec{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{k}}}} \left[a_{\vec{k}}^{(s)} u_s(\vec{k}) e^{ik\cdot x} + \left(c_{\vec{k}}^{(s)}\right)^\dagger v_s(\vec{k}) e^{-ik\cdot x} \right] \ . \tag{17}$$

   where the operators $a_{\vec{k}}^{(s)}, (a_{\vec{k}}^{(s)})^\dagger$ and $c_{\vec{k}}^{(s)}, (c_{\vec{k}}^{(s)})^\dagger$ satisfy the relations

$$\{a_{\vec{k}}^{(r)}, (a_{\vec{k}'}^{(s)})^\dagger\} = \{c_{\vec{k}}^{(r)}, (c_{\vec{k}'}^{(s)})^\dagger\} = \delta_{rs}(2\pi)^3\delta^{(3)}(\vec{k} - \vec{k}'), \tag{18}$$

$$\{a_{\vec{k}}^{(r)}, a_{\vec{k}'}^{(s)}\} = \{c_{\vec{k}}^{(r)}, c_{\vec{k}'}^{(s)}\} = 0 \ . \tag{19}$$

Problem Set 8

1. **Some proofs (21 points)**

   (a) From the definitions of (6), prove (9).

   Hint: use (8).

   Note: in lecture we derived the infinitesimal version of (9)*. Here you need to prove the finite version.*

   (b) Prove (10).

   (c) From Lorentz transformation of ψ, show that $\bar{\psi}\psi$ and $\bar{\psi}\gamma^\mu\psi$ transform under respectively as a scalar and a vector.

2. **Some identities (8 points)**

   Without using any explicit form of u_s and v_s, show: one of the relations in (13). In other words, show one of the following

$$u_r^\dagger(\vec{k})u_s(\vec{k}) = -\frac{iE}{m}\bar{u}^r(\vec{k})u_s(\vec{k}), \qquad v_r^\dagger(\vec{k})v_s(\vec{k}) = \frac{iE}{m}\bar{v}^r(\vec{k})v_s(\vec{k}) \ . \tag{20}$$

3. **Stress tensor and the Hamiltonian for the Dirac theory (21 points)**

   (a) The Dirac action is translationally invariant. Use the Noether procedure to construct the corresponding conserved currents $\Theta^{\mu\nu}$, i.e. the energy-momentum tensor.

   The "charge" density Θ^{00} for time translation is the energy density. Show that your Θ^{00} indeed coincides with the Hamiltonian density $\mathcal{H}$ we derived in lecture, i.e.

$$\Theta^{00} = \mathcal{H} = i\bar{\psi}(\gamma^i\partial_i - m)\psi \ . \tag{21}$$

   (b) Show that using the Dirac equations the Hamiltonian can be written as

$$H = i\int d^3x\, \psi^\dagger\partial_t\psi \tag{22}$$

   and express H in terms of $a_{\vec{k}}^{(s)}, (a_{\vec{k}}^{(s)})^\dagger$ and $c_{\vec{k}}^{(s)}, (c_{\vec{k}}^{(s)})^\dagger$ introduced in (17).

   (c) What is the vacuum energy density? (You can leave the answer as an integral). Discuss the differences with that for a scalar.

4. **Angular momentum operators (30 points)**

   The Dirac action (4) is Lorentz invariant.

   (a) Write down an infinitesimal Lorentz transformation for ψ.

(b) Use the Noether procedure to construct the conserved *charges* $M^{\mu\nu}$ associated with Lorentz transformations, and show that $M^{\mu\nu}$ can be written as a sum of a "spin" part $S^{\mu\nu}$ and an "orbital angular momentum" part $L^{\mu\nu}$, i.e.

$$M^{\mu\nu} = S^{\mu\nu} + L^{\mu\nu} \ . \tag{23}$$

Show that the orbital part $L^{\mu\nu}$ has the same form as that of a scalar, i.e.

$$L^{\mu\nu} = \int d^3x \left(x^\mu \Theta^{0\nu} - x^\nu \Theta^{0\mu}\right) \tag{24}$$

where $\Theta^{\mu\nu}$ is the energy-momentum tensor you obtained in 2(a). Show that the spin part $S^{\mu\nu}$ can be written as

$$S^{\mu\nu} = -\int d^3x \, \psi^\dagger \Sigma^{\mu\nu} \psi \ . \tag{25}$$

(c) Express the $S^{\mu\nu}$ in terms of $a_{\vec{k}}^{(s)}, (a_{\vec{k}}^{(s)})^\dagger$ and $c_{\vec{k}}^{(s)}, (c_{\vec{k}}^{(s)})^\dagger$.

(d) From the expression you obtained from part (c) for S^{ij}, keep only the time-independent part, which we will denote as $\tilde{S}^{ij}$. Define

$$\vec{J}^2 \equiv \frac{1}{2} \tilde{S}^{ij} \tilde{S}_{ij} \ . \tag{26}$$

Show that the one-particle states constructed by acting $(a_{\vec{k}}^{(s)})^\dagger$ and $(c_{\vec{k}}^{(s)})^\dagger$ with $\vec{k} = 0$ (i.e. in the rest frame) on the vacuum are eigenstates of $\vec{J}^2$ with eigenvalues corresponding to that of a spin-$\frac{1}{2}$ particle.

Note: the reason that we need not worry about the time-dependent part of S^{ij} is that the part will be canceled by a contribution from L^{ij}. M^{ij} is conserved: so the time dependent parts of S^{ij} and L^{ij} have to cancel each other.

Note: the restriction to the rest frame is important here as S_{ij} is not covariant under a Lorentz transformation. A covariant version of it is called the Pauli-Lubanski pseudovector, which we will not go into here.

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