Quantum Field Theory I (8.323) Spring 2023
Assignment 10

Apr. 18, 2023

• Please remember to put your name at the top of your paper.

Readings

• Peskin & Schroeder Chap. 3
• Peskin & Schroeder Chap. 9.5
• Peskin & Schroeder Chap. 4.7

Notes: conventions and some useful formulae

1. Conventions of $\gamma$ matrices:

\[ \{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta^{\mu\nu} \]  
and

\[ (\gamma^{\mu})^\dagger = \gamma^{0}\gamma^{\mu}\gamma^{0}. \]

2. The Dirac equation has the form

\[ (\gamma^{\mu}\partial_{\mu} - m)\psi = 0 \]

and the action is given by

\[ S = -i \int d^4x \bar{\psi}(\partial - m)\psi. \]

3. $u_{s}(\vec{k})e^{ik\cdot x}$ and $v_{s}(\vec{k})e^{-ik\cdot x}$, $s = 1, 2$ denote respectively a basis of positive and negative energy solutions to the Dirac equation, with $k^2 = -m^2$.

4. We normalize $u_{s}(\vec{k})$ and $v_{s}(\vec{k})$ as

\[ \bar{u}_{r}(\vec{k})u_{s}(\vec{k}) = 2mi\delta_{rs}, \quad \bar{v}_{r}(\vec{k})v_{s}(\vec{k}) = -2mi\delta_{rs}. \]

$u_{s}(\vec{k})$ and $v_{s}(\vec{k})$ are orthogonal

\[ \bar{u}_{r}(\vec{k})v_{s}(\vec{k}) = 0, \quad \bar{v}_{r}(\vec{k})u_{s}(\vec{k}) = 0. \]
5. With normalization (5), we have
\[ u_r^\dagger(\vec k)u_s(\vec k) = 2E\delta_{rs}, \quad v_r^\dagger(\vec k)v_s(\vec k) = 2E\delta_{rs}, \] (7)
and the orthogonal relations (6) can also be written as
\[ u_r^\dagger(\vec k)v_s(\vec k) = 0, \quad v_r^\dagger(\vec k)u_s(\vec k) = 0. \] (8)
These relations are valid for any choices of basis and any representation of gamma matrices once the normalizations are fixed as in (5).

6. An operator solution \( \psi(x) \) to the Dirac equation can be expanded as
\[
\psi(x) = \int \frac{d^3\vec k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} \left[ a_k^{(s)}(\vec k)u_s(\vec k)e^{ik\cdot x} + \left( c_k^{(s)}\right)^\dagger v_s(\vec k)e^{-ik\cdot x} \right].
\] (9)
where the operators \( a_k^{(s)}, (a_k^{(s)})^\dagger \) and \( c_k^{(s)}, (c_k^{(s)})^\dagger \) satisfy the relations
\[
\{ a_k^{(r)}, (a_k^{(s)})^\dagger \} = \{ c_k^{(r)}, (c_k^{(s)})^\dagger \} = \delta_{rs}(2\pi)^3\delta^{(3)}(\vec k - \vec k'), \quad (10)
\]
\[
\{ a_k^{(r)}, a_k^{(s)} \} = \{ c_k^{(r)}, c_k^{(s)} \} = 0. \quad (11)
\]

**Problem Set 10**

1. **Chiral symmetry (15 points)**

   Consider the Dirac action with \( m = 0 \).

   (a) Show that the action is invariant under transformations
   \[
   \psi \rightarrow e^{i\alpha \gamma^5} \psi.
   \] (12)

   (b) Construct the Noether current for the above symmetry.

   (c) Find how the mass term \( m\bar{\psi}\psi \) transforms under (12). Is it invariant?

2. **Quantizing the theory of Majorana fermions (25 points)**

   Consider the theory of Majorana fermions discussed in Pset 9, written in terms of a two-component complex spinor \( \psi_L \)
   \[
   \mathcal{L}_L = i\psi_L^\dagger \sigma^\mu \partial_\mu \psi_L - \frac{m}{2} (\psi_L^T \sigma^2 \psi_L + \psi_L^\dagger \sigma^2 \psi_L^*). \] (13)
where \( \psi^T \) denotes the transpose of \( \psi \), and \( \sigma^\mu = (1, \vec{\sigma}) \).
(a) Write down the equal time canonical quantization relations.

(b) Write down the classical equations of motion. In momentum space a general solution can be written as

$$\psi_L(x) = u(p)e^{ip\cdot x} + v(p)e^{-ip\cdot x}. \quad (14)$$

Using the above notations, write down a complete basis of solutions in the rest frame (i.e. $\vec{p} = 0$).

(c) Verify the following expressions give a complete basis of solutions for general $p$ (below $\vec{\sigma} = (1, -\vec{\sigma})$)

$$u_s(p) = \sqrt{-p \cdot \vec{\sigma}} \zeta_s \quad (15)$$

$$v_s(p) = -\sqrt{-p \cdot \vec{\sigma}} \sigma^2 \zeta_s \quad (16)$$

where $\zeta_s, s = \pm$ are respectively eigenvectors of $\sigma^3$ with eigenvalues $\pm$.

(d) Write down the mode expansion for quantum operator $\psi_L$.

(e) Define the vacuum and construct the single-particle states (properly normalized). Discuss the differences between the particles in this theory and those of the Dirac theory.

3. Gaussian integrals for Grassman variables (16 points)

Show that

$$\int \prod_{i=1}^{N} (d\theta_i^* d\theta_i) \ e^{-\theta_i^* A_{ij} \theta_j} = \det A \quad (17)$$

and

$$\int \prod_{i=1}^{N} (d\theta_i^* d\theta_i) \ \theta_k \theta_l^* e^{-\theta_i^* A_{ij} \theta_j} = \det A(A^{-1})_{kl}. \quad (18)$$

4. Yukawa theory (24 points)

Consider the Yukawa theory discussed in lecture

$$\mathcal{L} = -\frac{1}{2} (\partial \phi)^2 - \frac{1}{2} m^2 \phi^2 - i \bar{\psi}(\not\! \! \! p - m) \psi - g \phi \bar{\psi} \psi. \quad (19)$$

Denote the propagator of a $\phi$ particle by a dashed line and that of $\psi$ by a solid line (with arrow). We will call $p$ the particle excitation of $\psi$ and $\bar{p}$ and the anti-particle excitation of $\bar{\psi}$.

(a) Consider the process

$$\bar{p} + \bar{p} \rightarrow \bar{p} + \bar{p} \quad (20)$$

Draw the lowest order Feynman diagrams and write down the corresponding scattering amplitude. Take the initial and final states have momenta and polarizations $(p_1, s_1), (p_2, s_2)$ and $(p'_1, s'_1), (p'_2, s'_2)$ respectively.
(b) Consider the process
\[ p + \bar{p} \rightarrow \phi + \phi \] (21)
Draw the lowest order Feynman diagrams and write down the corresponding scattering amplitude. Take the initial and final states have momenta and polarizations \((p_1, s_1), (p_2, s_2)\) and \(p'_1, p'_2\) respectively.

(c) Consider the process
\[ p + \phi \rightarrow p + \phi \] (22)
Draw the lowest order Feynman diagrams and write down the corresponding scattering amplitude. Take the initial and final states have momenta and polarizations \((p_1, s_1), p_2\) and \((p'_1, s'_1), p'_2\) respectively.