8.323 Problem Set 5 Solutions

March 14, 2023

Question 1: A Useful Formula, and the Path Integral in Phase Space (20 points)

(a) Derive the following equations:

$$\left\langle x_{i+1} \left| e^{-i\frac{\hat{p}^2}{2m}\Delta t} e^{-i\Delta t V(\hat{x})} \right| x_i \right\rangle = \int đp_i \exp\left[-i\Delta t \frac{p_i^2}{2m} - i\Delta t V(x_i) + ip_i(x_{i+1} - x_i)\right]$$
$$= \sqrt{\frac{m}{2\pi i \Delta t}} \exp\left[\frac{im\Delta t}{2}\left(\frac{x_{i+1} - x_i}{\Delta t}\right)^2 - i\Delta t V(x_i)\right]$$

We start by deriving the first equation:

$$\begin{aligned}
\left\langle x_{i+1} \left| e^{-i\frac{\hat{p}^2}{2m}\Delta t} e^{-i\Delta t V(\hat{x})} \right| x_i \right\rangle &= \int dp_i \langle x_{i+1}|e^{-i\frac{\hat{p}^2}{2m}\Delta t}|p_i\rangle\langle p_i|e^{-i\Delta t V(\hat{x})}|x_i\rangle \\
&= \int dp_i e^{-i\frac{p_i^2}{2m}\Delta t} e^{-i\Delta t V(x_i)} \langle x_{i+1}|p_i\rangle\langle p_i|x_i\rangle \\
&= \int đp_i e^{-i\frac{p_i^2}{2m}\Delta t} e^{-i\Delta t V(x_i)} e^{ip_i x_{i+1}} e^{-ip_i x_i} \\
&= \int đp_i \exp\left[-i\Delta t \frac{p_i^2}{2m} - i\Delta t V(x_i) + ip_i(x_{i+1} - x_i)\right]
\end{aligned}$$

To get the second equation from this, we complete the square and compute the Gaussian integral, or equivalently use the identity $\int dx e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{b^2/4a}$. Hence,

$$\begin{aligned}
\left\langle x_{i+1} \left| e^{-i\frac{\hat{p}^2}{2m}\Delta t} e^{-i\Delta t V(\hat{x})} \right| x_i \right\rangle &= \frac{1}{2\pi}\sqrt{\frac{2\pi m}{i\Delta t}} \exp\left[-\frac{(x_{i+1} - x_i)^2}{2i\Delta t/m}\right] e^{-i\Delta t V(x_i)} \\
&= \sqrt{\frac{m}{2\pi i \Delta t}} \exp\left[\frac{im\Delta t}{2}\left(\frac{x_{i+1} - x_i}{\Delta t}\right)^2 - i\Delta t V(x_i)\right]
\end{aligned}$$

(b) Use the first equation in (a) to derive the following:

$$\langle x_a, t_a | x_b, t_b\rangle = \int_{x(t_b)=x_b}^{x(t_a)=x_a} Dx(t)Dp(t) \exp\left[i\int_{t'}^{t} dt (p\dot{x} - H)\right]$$

The integration in this expression should be understood as

$$\int_{x(t_b)=x_b}^{x(t_a)=x_a} Dx(t)Dp(t) = \lim_{N\to\infty} \int \frac{dp_0}{2\pi} \int \frac{dx_1 dp_1}{2\pi} \cdots \int \frac{dx_{N-1}dp_{N-1}}{2\pi}$$

where we again divide the interval $[t_b, t_a]$ into N segments, with $t_0 = t_b$, $t_N = t_a$.

We denote the left-hand side of the equation in (a) by M_i. Now we do the same trick from class to derive the path-integral:

$$
\begin{aligned}
\langle x_a, t_a | x_b, t_b \rangle &= \langle x_a | e^{-i\hat{H}(t_a - t_b)} | x_b \rangle = \langle x_a | (e^{-i\hat{H}\Delta t})^N | x_b \rangle \\
&= \int dx_1 \cdots dx_{N-1} \langle x_a | e^{-i\hat{H}\Delta t} | x_{N-1} \rangle \langle x_{N-1} | \cdots | x_1 \rangle \langle x_1 | e^{-i\hat{H}\Delta t} | x_b \rangle \\
&= \int dx_1 \cdots dx_{N-1} M_{N-1} \cdots M_0 \\
&= \int dx_1 \cdots dx_{N-1} đp_0 \cdots đp_{N-1} \prod_{i=0}^{N-1} \exp \left[-i\Delta t \frac{p_i^2}{2m} - i\Delta t V(x_i) + ip_i(x_{i+1} - x_i) \right] \\
&= \int_{x(t_b)=x_b}^{x(t_a)=x_a} Dx(t) Dp(t) \exp \left[i \sum_{i=0}^{N-1} \Delta t \left(p_i \frac{x_{i+1} - x_i}{\Delta t} - \frac{p_i^2}{2m} - V(x_i) \right) \right] \\
&= \int_{x(t_b)=x_b}^{x(t_a)=x_a} Dx(t) Dp(t) \exp \left[i \int_{t_b}^{t_a} dt (p\dot{x} - H) \right]
\end{aligned}
$$

In the second line we insert the identity $N-1$ times, so that we can use the formula from (a) for each matrix element in the 4th line. In the 5th line the product of exponentials becomes a sum of exponents, which in the continuum case reduces to the integral in the last line.

Question 2: The Schrödinger Equation, Rederived (20 points)
The wavefunction $\psi(t,x)$ for a system at time t can be obtained from that at time t' by

$$\psi(t,x) = \int dx' K(x,t;x',t')\psi(x',t')$$

for the propagator given by

$$K(x,t;x',t') = \langle x,t|x',t'\rangle = \int_{x(t_b)=x_b}^{x(t_a)=x_a} Dx(t) \exp\left[i\int_{t'}^{t} dt L(\dot{x},x)\right]$$

Show that $\psi(t,x)$ satisfies the Schrödinger equation,

$$i\partial_t\psi(t,x) = -\frac{1}{2m}\partial_x^2\psi(t,x) + V(x)\psi(t,x)$$

We therefore consider the wavefunction after an infinitesimal time step δt:

$$\psi(t+\delta t,x) = \int dy K(t+\delta t,x;t,y)\psi(t,y)$$

Equating both sides of this equation to order δ will lead to the Schrödinger equation.
The left hand side is simple:

$$\text{LHS} = \psi(t+\delta t,x) = \psi(t,x) + \delta t\partial_t\psi(t,x) + \mathcal{O}(\delta t^2)$$

To expand the right hand side, note for an infinitesimal δt that the propagator $K(t+\delta t,x;t,y)$ can be written as a single infinitesimal time step:

$$K(t+\delta t,x;t,y) = \left(\frac{m}{2\pi i\delta t}\right)^{1/2} \exp\left[i\delta t\left(\frac{m}{2}\left(\frac{x-y}{\delta t}\right)^2 - V(y)\right)\right]$$

Therefore we expand:

$$\begin{aligned}
\text{RHS} &= \int dy K(t+\delta t,x;t,y)\psi(t,y) \\
&= \left(\frac{m}{2\pi i\delta t}\right)^{1/2} \int dy e^{i\delta t\left(\frac{m}{2}\left(\frac{x-y}{\delta t}\right)^2 - V(y)\right)}\psi(t,y) \\
&= \left(\frac{m}{2\pi i}\right)^{1/2} \int du e^{i\left(\frac{m}{2}u^2 - \delta t V(x+u\sqrt{\delta t})\right)}\psi(t,x+u\sqrt{\delta t}) \\
&= \left(\frac{m}{2\pi i}\right)^{1/2} \int du e^{i\frac{m}{2}u^2}(1-\delta t V(x))\left(\psi(t,x) + \sqrt{\delta t}\partial_x\psi(t,x)u + \frac{\delta t}{2}\partial^2 x\psi(t,u)u^2\right) + \mathcal{O}(\delta t^2) \\
&= \psi(t,x) + \delta t\left(-iV(x)\psi(t,x) + \frac{i}{2m}\partial_x^2\psi(t,x)\right) + \mathcal{O}(\delta t^2)
\end{aligned}$$

In the third line we have made the substitution $u\sqrt{\delta t} = y - x$, and in the 4th line we expand each term to order δt, neglecting higher order terms. In the subsequent line we perform the Gaussian integrals over u, noting that the order $(\delta t)^{1/2}$ term becauyse the integral is odd.

Setting the 2 sides equal, we have the Schrödinger equation as desired:

$$i\partial_t\psi(t,x) = \left(-\frac{1}{2m}\partial_x^2 + V(x)\right)\psi(t,x)$$

Question 3: The Free Particle (20 points)

For a free particle, i.e. $V(x) = 0$, perform explicitly the integrals over x_i, $i = 1, \dots, N-1$ to show that

$$K(x_a, t_a; x_b, t_b) = \left(\frac{m}{2\pi i(t_a - t_b)}\right)^{1/2} \exp\left[\frac{im(x_a - x_b)^2}{2(t_a - t_b)}\right]$$

We evaluate the integrals in order, starting with the one over x_1. We denote $x_0 = x_b$ and $x_N = x_a$.

$$\begin{aligned}
K(x_a, t_a; x_b, t_b) &= \lim_{N\to\infty} \left(\frac{m}{2\pi i \Delta t}\right)^{N/2} \int dx_1 \cdots dx_{N-1} e^{i\frac{m}{2\Delta t}((x_N - x_{N-1})^2 + \cdots + (x_2 - x_1)^2 + (x_1 - x_0)^2)} \\
&= \lim_{N\to\infty} \left(\frac{m}{2\pi i \Delta t}\right)^{N/2} \left(\frac{i\pi\Delta t}{m}\right)^{1/2} \int dx_2 \cdots dx_{N-1} e^{i\frac{m}{2\Delta t}((x_N - x_{N-1})^2 + \cdots + (x_3 - x_2)^2 + \frac{1}{2}(x_2 - x_0)^2)} \\
&= \lim_{N\to\infty} \left(\frac{m}{2\pi i \Delta t}\right)^{\frac{N-1}{2}} \frac{1}{\sqrt{2}} \int dx_2 \cdots dx_{N-1} e^{i\frac{m}{2\Delta t}((x_N - x_{N-1})^2 + \cdots + (x_3 - x_2)^2 + \frac{1}{2}(x_2 - x_0)^2)} \\
&= \lim_{N\to\infty} \left(\frac{m}{2\pi i \Delta t}\right)^{\frac{N-1}{2}} \frac{2}{\sqrt{2}} \left(\frac{i\pi\Delta t}{3m}\right)^{1/2} \int dx_3 \cdots dx_{N-1} e^{i\frac{m}{2\Delta t}((x_N - x_{N-1})^2 + \cdots + \frac{1}{3}(x_3 - x_0)^2)} \\
&= \lim_{N\to\infty} \left(\frac{m}{2\pi i \Delta t}\right)^{\frac{N-2}{2}} \frac{1}{\sqrt{3}} \int dx_3 \cdots dx_{N-1} e^{i\frac{m}{2\Delta t}((x_N - x_{N-1})^2 + \cdots + (x_4 - x_3)^2 + \frac{1}{3}(x_3 - x_0)^2)} \\
&= \cdots = \lim_{N\to\infty} \left(\frac{m}{2\pi i N \Delta t}\right)^{1/2} e^{i\frac{m}{2N\Delta t}(x_a - x_b)^2} \\
&= \sqrt{\frac{m}{2\pi i N(t_a - t_b)}} e^{i\frac{m}{2}\frac{(x_a - x_b)^2}{t_a - t_b}}
\end{aligned}$$

Question 4: The Path Integral for a Free Particle, Revisited (20 points)
In this problem we evaluate the path integral for a free particle using a different method than Problem 3. For simplicity, take $x_a = x_b = 0, t_b = 0, t_a = T$. As discussed in class, K is a Gaussian path-integral of form:

$$K(0,T;0,0) = \int_{x(0)=0}^{x(T)=0} Dx(t) \exp\left[\frac{i}{2}\int dtdt' x(t)A(t,t')x(t')\right]$$

for some differential operator A.
(a) Write down the explicit expression for A.
The action for a free particle is given by

$$\begin{aligned} S[x(t)] &= \int dt \frac{m}{2}\partial_t x \partial_t x = -\frac{m}{2}\int dt x \partial_t^2 x = -\frac{m}{2}\int dtdt' x(t')\delta(t-t')\partial_t^2 x(t) \\ &= \frac{1}{2}\int dtdt' x(t')A(t',t)x(t), \qquad A(t',t) = -m\delta(t-t')\partial_t^2 \end{aligned}$$

Then, we write the propagator as

$$K(0,T;0,0) = \int_{x(0)=0}^{x(T)=0} Dx e^{iS} = \int_{x(0)=0}^{x(T)=0} Dx \exp\left[\frac{i}{2}\int_0^T dtdt' x(t')A(t',t)x(t)\right]$$

for the differential operator $A(t',t) = -m\delta(t-t)\partial_t^2$.

(b) Find all the eigenvalues of A. Show that the determinant of A can be written as

$$\det A = \prod_{n=1}^{\infty} m\frac{n^2\pi^2}{T^2}$$

The eigenvectors of a second order derivative operator are precisely exponentials, $x(t) = f_\lambda(t) = e^{i\lambda t}$. We further need our eigenvectors to satisfy the boundary conditions $x(0) = x(T) = 0$, since the endpoints of our trajectory are fixed. Hence, a complete set of eigenvectors are given by sine functions, with momenta integer multiples of π/T. Without loss of generality we can restrict to $n > 0$.

$$f_{\lambda_n}(t) = \sqrt{\frac{2}{T}}\sin\left(\frac{n\pi t}{T}\right)$$

Furthermore,

$$\int dtA(t',t)f_{\lambda_n}(t) = -m\int dt\delta(t-t')\sqrt{\frac{2}{T}}\partial_t^2\sin\left(\frac{n\pi t}{T}\right) = m\frac{n^2\pi^2}{T^2}\sqrt{\frac{2}{T}}\sin\left(\frac{n\pi t}{T}\right) = \lambda_n f_{\lambda_n}(t)$$

for the eigenvalue $\lambda_n = m\frac{n^2\pi^2}{T^2}$. In this basis A is diagonal: $A_{mn} = m\frac{n^2\pi^2}{T^2}\delta_{nm}$. Finally, the determinant of an operator is obtained by multiplying all of its eigenvalues, giving us the desired expression:

$$\det A = \prod_{n=1}^{\infty} m\frac{n^2\pi^2}{T^2}$$

(c) Since our propagator is Gaussian, it can be evaluated as

$$K(0,T;0,0) = \frac{C}{\sqrt{\det A}}$$

where C is some constant. By comparing this equation with the solution to Problem 3, show that the consistency of the 2 approaches requires

$$\frac{C}{\sqrt{\det A}} = \left(\frac{m}{2\pi i T}\right)^{1/2}$$

We substitute $x_a = x_b = 0, t_a = T, t_b = 0$ into the result in Problem 3:

$$K(0,T;0,0) = \left(\frac{m}{2\pi i T}\right)^{1/2} \exp\left[\frac{im0^2}{2T}\right] = \left(\frac{m}{2\pi i T}\right)^{1/2}$$

It therefore follows immediately from our work in (b) that

$$K(0,T;0,0) = \left(\frac{m}{2\pi i T}\right)^{1/2} = \frac{C}{\sqrt{\det A}} = \frac{CT}{\pi\sqrt{m}} \prod_{n=1}^{\infty} \frac{1}{n}$$

8.323 Relativistic Quantum Field Theory I
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