8.323 Problem Set 11 Solutions

May 2, 2023

Question 1: Quantization of Maxwell Theory in the Lorentz Gauge (50 points)

In the Lorentz gauge we consider the action

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{\xi}{2}(\partial_\mu A^\mu)^2$$

where ξ is an arbitrary real parameter (and different ξ's give equivalent theories). It is convenient to take $\xi = 1$, in which case

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu$$

The complete set of solutions to the operator equations are

$$A_\mu(x) = \int \frac{đ^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}} \sum_{\alpha=0}^{3} \left[\mathcal{E}_\mu^{(\alpha)} a_\mathbf{k}^{(\alpha)} e^{ik\cdot x} + \mathcal{E}_\mu^{(\alpha)*} a_\mathbf{k}^{(\alpha)\dagger} e^{-ik\cdot x}\right]$$

where $\omega_\mathbf{k} = |\mathbf{k}|$, and $k^\mu = (|\mathbf{k}|, \mathbf{k})$. The polarization vectors $\mathcal{E}_\mu^{(\alpha)}$ are defined by

$$\mathcal{E}_\mu^{(0)} = (1, \mathbf{0}), \qquad \mathcal{E}_\mu^{(3)} = (0, \mathbf{k}/|\mathbf{k}|), \qquad \mathcal{E}_\mu^{(1,2)} = (0, \boldsymbol{\epsilon}_{1,2}), \qquad \boldsymbol{\epsilon}_{1,2} \cdot \mathbf{k} = 0$$

where $\boldsymbol{\epsilon}_{1,2}$ are orthogonal unit-norm spatial vectors.

With the canonical commutation relations

$$[A_\mu(t, \mathbf{x}), \pi^\nu(t, \mathbf{x}')] = i\delta_\mu^\nu \delta^{(3)}(\mathbf{x} - \mathbf{x}'), \qquad \pi^\mu = \partial_0 A^\mu$$

we find that

$$[a_\mathbf{k}^{(\alpha)}, a_{\mathbf{k}'}^{(\beta)\dagger}] = \eta^{\alpha\beta}(2\pi)^3\delta^{(3)}(\mathbf{k} - \mathbf{k}')$$

with all other commutators vanishing. We define the vacuum as

$$a_\mathbf{k}^{(\alpha)}|0\rangle = 0, \qquad \forall \alpha, \mathbf{k}$$

and the 'big' Hilbert space is defined as

$$\mathcal{H}_{\text{big}} = \{|\psi\rangle \text{ obtained by acting } a_\mathbf{k}^{(\alpha)\dagger} \text{ on } |0\rangle\}$$

As discussed in lecture, $\mathcal{H}_{\text{big}}$ contains states with negative norms, and thus unphysical states. Indeed $\mathcal{H}_{\text{big}}$ follows from the gauge-fixed Lagrangian, which by itself is not Maxwell theory.

To obtain the Maxwell theory, we still need to impose $\partial_\mu A^\mu = 0$ and get rid of the residual gauge freedom. We will see in this problem that by imposing $\partial_\mu A^\mu = 0$ we get rid of the negative-norm unphysical states in $\mathcal{H}_{\text{big}}$. But this is not enough: we will observe that some states possess zero norm, which can be attributed to the presence of residual gauge freedom. By eliminating these null states, we obtain the physical Hilbert space, which contains only 2 transverse massless degrees of freedom rather than 4.

The enforcement of $\partial_\mu A^\mu$ at the quantum level is subtle. Remember that in the Coulomb gauge, classically we apply the gauge condition $\nabla_i A_i = 0$ as part of the equations of motion, which becomes an operator equation at the quantum level. In the Lorentz gauge, classically we only need to impose 'boundary conditions' to ensure that the equation $\partial^2(\partial_\mu A^\mu) = 0$ has the trivial solution $\partial_\mu A^\mu = 0$. This implies that, at the quantum level, we cannot enforce $\partial_\mu A^\mu = 0$ as an operator equation. Instead, we will have to do something weaker, imposing a variant of it as a condition on the physical states.
(a) Calculate

$$[A_0(t, \mathbf{x}), \partial_\mu A^\mu(t, \mathbf{x}')]$$

and from your result explain why we cannot impose $\partial_\mu A^\mu = 0$ as an operator equation.
In the mode expansion for A_0, only the $\alpha = 0$ components survive, since $\mathcal{E}^{(0)}$ is the only polarization with a time-like component.

$$A_0 = \int \frac{\text{đ}^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}} \left[a_\mathbf{k}^{(0)} e^{ik\cdot x} + a_\mathbf{k}^{(0)\dagger} e^{-ik\cdot x} \right]$$

Next, we compute $\partial_\mu A^\mu$ in the mode expansion.

$$\begin{aligned}
\partial_\mu A^\mu &= i \int \frac{\text{đ}^3\mathbf{k}}{\sqrt{2\omega_\mathbf{k}}} \sum_{\alpha=0}^{3} \left[(k\cdot\mathcal{E}^{(\alpha)}) a_\mathbf{k}^{(\alpha)} e^{ik\cdot x} - (k\cdot\mathcal{E}^{(\alpha)*}) a_\mathbf{k}^{(\alpha)\dagger} e^{-ik\cdot x} \right] \\
&= i \int \text{đ}^3\mathbf{k} \sqrt{\frac{\omega_\mathbf{k}}{2}} \left[(a_\mathbf{k}^{(0)} + a_\mathbf{k}^{(3)}) e^{ik\cdot x} - (a_\mathbf{k}^{(0)\dagger} + a_\mathbf{k}^{(3)\dagger}) e^{-ik\cdot x} \right]
\end{aligned}$$

where we use $k\cdot\mathcal{E}^{(0)} = \omega_\mathbf{k}$, $k\cdot\mathcal{E}^{(1,2)} = 0$, $k\cdot\mathcal{E}^{(3)} = \omega_\mathbf{k}$ by using the explicit form of the polarizations. Therefore,

$$\begin{aligned}
[A_0(t,\mathbf{x}), \partial_\mu A^\mu(t,\mathbf{x}')] &= \frac{i}{2} \int \frac{\text{đ}^3\mathbf{k}\text{đ}^3\mathbf{k}'}{\sqrt{\omega_\mathbf{k}/\omega_{\mathbf{k}'}}} \left[a_\mathbf{k}^{(0)} e^{ik\cdot x} + a_\mathbf{k}^{(0)\dagger} e^{-ik\cdot x}, (a_{\mathbf{k}'}^{(0)} + a_{\mathbf{k}'}^{(3)}) e^{ik'\cdot x'} - (a_{\mathbf{k}'}^{(0)\dagger} + a_{\mathbf{k}'}^{(3)\dagger}) e^{-ik'\cdot x'} \right] \\
&= -\frac{i}{2} \int \frac{\text{đ}^3\mathbf{k}\text{đ}^3\mathbf{k}'}{\sqrt{\omega_\mathbf{k}/\omega_{\mathbf{k}'}}} \left([a_\mathbf{k}^{(0)}, a_{\mathbf{k}'}^{(0)\dagger}] e^{i(k\cdot x - k'\cdot x')} + [a_{\mathbf{k}'}^{(0)}, a_\mathbf{k}^{(0)\dagger}] e^{i(-k\cdot x + k'\cdot x')} \right) \\
&= -i \int \text{đ}^3\mathbf{k}\text{đ}^3\mathbf{k}' \sqrt{\frac{\omega_{\mathbf{k}'}}{\omega_\mathbf{k}}} (-(2\pi)^3 \delta^{(3)}(\mathbf{k} - \mathbf{k}')) e^{i(k\dot{x} - k'\cdot x')} \\
&= i \int \text{đ}^3\mathbf{k} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} = i(2\pi)^3 \delta^3(\mathbf{x} - \mathbf{x}')
\end{aligned}$$

In the 3rd equality, we used that the two terms in line 2 are equal, upon change of variables $\mathbf{k} \leftrightarrow \mathbf{k}'$.

We see that the commutator is non-vanishing. This prevents us from imposing $\partial_\mu A^\mu = 0$ as an operator equation, which would imply the commutator vanishes identically.

(b) We also cannot impose that 'physical states' satisfy

$$\partial_\mu A^\mu |\psi\rangle = 0$$

as $\partial_\mu A^\mu |0\rangle \neq 0$, and we want to keep the vacuum as physical. So, to define physical states we need a weaker condition. It turns out the condition eliminating all negative norm states while keeping $|0\rangle$ is

$$\partial^\mu A_\mu^{(-)} |\psi\rangle = 0$$

where $A_\mu^{(-)}$ denotes the annihilation part of A_μ. We now impose that the set of physical states $\mathcal{H}_{\text{small}} \subseteq \mathcal{H}_{\text{big}}$ satisfies this equation. Show that there is no negative-norm state in $\mathcal{H}_{\text{small}}$.

We show that any negative-norm state $|\psi\rangle \in \mathcal{H}_{\text{big}}$ does not satisfy the defining condition $\partial^\mu A_\mu^{(-)}|\psi\rangle = 0$, and thus does not belong to $\mathcal{H}_{\text{small}}$.

From the calculation in part (a), we have:

$$\partial^\mu A_\mu^{(-)} = i \int đ^3\mathbf{k} \sqrt{\frac{\omega_{\mathbf{k}}}{2}} (a_{\mathbf{k}}^{(0)} + a_{\mathbf{k}}^{(3)}) e^{ik\cdot x}$$

The negative norm states in $\mathcal{H}_{\text{big}}$ come from applying instances of the raising operator $a_{\mathbf{p}}^{(0)\dagger}$. We prove the desired result for the negative norm states $|\psi\rangle = a_{\mathbf{p}}^{(0)\dagger}|\chi\rangle$, where $|\chi\rangle$ is a positive norm state. (Note: this is a sufficient, but not necessary condition for negative norm state, as a negative norm state can be obtained from a linear combination of negative, null, and positive norm states. Taking these into account is more difficult, and is more or less equivalent to doing the rest of the problem. We therefore defer this to part (g)). We compute

$$\begin{aligned}
\partial^\mu A_\mu^{(-)}|\psi\rangle &= i \int đ^3\mathbf{k} \sqrt{\frac{\omega_{\mathbf{k}}}{2}} (a_{\mathbf{k}}^{(0)} + a_{\mathbf{k}}^{(3)}) e^{ik\cdot x} a_{\mathbf{p}}^{(0)\dagger}|\chi\rangle \\
&= -i\sqrt{\frac{\omega_{\mathbf{p}}}{2}} e^{ip\cdot x}|\chi\rangle + i \int đ^3\mathbf{k} \sqrt{\frac{\omega_{\mathbf{k}}}{2}} e^{ik\cdot x} a_{\mathbf{p}}^{(0)\dagger} a_{\mathbf{k}}^{(3)}|\chi\rangle
\end{aligned}$$

This cannot vanish, as the first term is non-zero, and the second term is orthogonal to the first.

(c) Show that

$$\langle\psi'|\partial_\mu A^\mu|\psi\rangle = 0, \qquad \forall|\psi\rangle, |\psi'\rangle \in \mathcal{H}_{\text{small}}$$

That is, $\partial_\mu A^\mu$ has zero matrix element among states in $\mathcal{H}_{\text{small}}$.

Note from the mode expansion in (b) that

$$\partial_\mu A^\mu = i \int đ^3\mathbf{k} \sqrt{\frac{\omega_{\mathbf{k}}}{2}} \left[(a_{\mathbf{k}}^{(0)} + a_{\mathbf{k}}^{(3)}) e^{ik\cdot x} - (a_{\mathbf{k}}^{(0)\dagger} + a_{\mathbf{k}}^{(3)\dagger}) e^{-ik\cdot x} \right] = \partial^\mu A_\mu^{(-)} - \partial^\mu A_\mu^{(-)\dagger}$$

Therefore, for $|\psi\rangle, |\psi'\rangle \in \mathcal{H}_{\text{small}}$ we have

$$\langle\psi'|\partial^\mu A_\mu|\psi\rangle = \langle\psi'|\partial^\mu A_\mu^{(-)}\psi\rangle - \langle\partial^\mu A_\mu^{(-)}\psi'|\psi\rangle = 0 - 0 = 0$$

(d) Introduce

$$b_{\mathbf{k}}^{(\pm)} = \frac{1}{\sqrt{2}}(a_{\mathbf{k}}^{(3)} \pm a_{\mathbf{k}}^{(0)}), \qquad b_{\mathbf{k}}^{(\pm)\dagger} = \frac{1}{\sqrt{2}}(a_{\mathbf{k}}^{(3)\dagger} \pm a_{\mathbf{k}}^{(0)\dagger})$$

Show that the physical state condition can be written as

$$b_{\mathbf{k}}^{(+)}|\psi\rangle = 0$$

From the mode expansion in (b), the physical state condition is

$$\partial^\mu A_\mu^{(-)}|\psi\rangle = i \int đ^3\mathbf{k} \sqrt{\frac{\omega_{\mathbf{k}}}{2}} e^{ik\cdot x} (a_{\mathbf{k}}^{(0)} + a_{\mathbf{k}}^{(3)})|\psi\rangle = i \int đ^3\mathbf{k} \sqrt{\omega_{\mathbf{k}}} e^{ik\cdot x} b_{\mathbf{k}}^{(+)}|\psi\rangle = 0$$

Since the states $b_{\mathbf{k}}^{(+)}|\psi\rangle$, $b_{\mathbf{k}'}^{(+)}|\psi\rangle$ are orthogonal for $\mathbf{k} \neq \mathbf{k}'$, for the above equation to hold we must have $b_{\mathbf{k}}^{(+)}|\psi\rangle = 0$ for all $\mathbf{k}$.

(e) We are not yet done, as $\mathcal{H}_{\text{small}}$ still contains zero-norm states. To see this, work out the commutators

$$[b_{\mathbf{k}}^{(+)}, b_{\mathbf{k}'}^{(+)\dagger}], \qquad [b_{\mathbf{k}}^{(-)}, b_{\mathbf{k}'}^{(-)\dagger}], \qquad [b_{\mathbf{k}}^{(+)}, b_{\mathbf{k}'}^{(-)\dagger}], \qquad [b_{\mathbf{k}}^{(-)}, b_{\mathbf{k}'}^{(+)\dagger}]$$

and show that $\mathcal{H}_{\text{small}}$ can also be described as

$$\mathcal{H}_{\text{small}} = \{\text{all states obtained by acting } a_{\mathbf{k}}^{(1)\dagger}, a_{\mathbf{k}}^{(2)\dagger}, b_{\mathbf{k}}^{(+)\dagger} \text{ on } |0\rangle\}$$

In other words, a physical state can have no $b_{\mathbf{k}}^{(-)\dagger}$ excitations.

The operators have the commutation relations

$$[b_{\mathbf{k}}^{(\pm)}, b_{\mathbf{k}'}^{(\pm)\dagger}] = \frac{1}{2}([a_{\mathbf{k}}^{(3)}, a_{\mathbf{k}'}^{(3)\dagger}] + [a_{\mathbf{k}}^{(0)}, a_{\mathbf{k}'}^{(0)\dagger}]) = 0$$
$$[b_{\mathbf{k}}^{(\pm)}, b_{\mathbf{k}'}^{(\mp)\dagger}] = \frac{1}{2}([a_{\mathbf{k}}^{(3)}, a_{\mathbf{k}'}^{(3)\dagger}] - [a_{\mathbf{k}}^{(0)}, a_{\mathbf{k}'}^{(0)\dagger}]) = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')$$

From (d), the condition for $|\psi\rangle \in \mathcal{H}_{\text{small}}$ is

$$b_{\mathbf{k}}^{(+)}|\psi\rangle = 0, \ \forall \mathbf{k}$$

Suppose $|\psi\rangle$ is generated by applying some number of $a_{\mathbf{k}}^{(1)\dagger}$, $a_{\mathbf{k}}^{(2)\dagger}$, $b_{\mathbf{k}}^{(+)\dagger}$ to the vacuum. By the commutation relations above, to compute $b_{\mathbf{k}}^{(+)}|\psi\rangle$ we may commute the instance of $b_{\mathbf{k}}^{(+)}$ to the very right, where it annihilates the vacuum to give 0.

Now suppose there are any instances of $b_{\mathbf{k}'}^{(-)\dagger}$ in generating $|\psi\rangle$ by applying creation operators to vacuum. Due to the commutation relations these serve as obstructions to commuting $b_{\mathbf{k}}^{(+)}$ to the very right, and doing so one picks up non-zero terms proportional to $\delta^3(\mathbf{k}-\mathbf{k}')$, which cannot cancel.

Combining these results, one finds the desired result that

$$\mathcal{H}_{\text{small}} = \{\text{all states obtained by acting } a_{\mathbf{k}}^{(1)\dagger}, a_{\mathbf{k}}^{(2)\dagger}, b_{\mathbf{k}}^{(+)\dagger} \text{ on } |0\rangle\}$$

(f) Show that any state $|\psi\rangle \in \mathcal{H}_{\text{small}}$ with non-zero $b_{\mathbf{k}}^{(+)\dagger}$ excitations has zero norm, and its overlap with any state in $\mathcal{H}_{\text{small}}$ is zero. Such states are called null states, and cannot have any physical significance.

Any state in $|\psi\rangle \in \mathcal{H}_{\text{small}}$ can be produced by acting some combination of $a_{\mathbf{k}}^{(1)\dagger}$, $a_{\mathbf{k}}^{(2)\dagger}$, $b_{\mathbf{k}}^{(+)\dagger}$ on $|0\rangle$. Taking the Hermitian conjugate, any state $\langle\psi'| \in \mathcal{H}_{\text{small}}$ can be produced by acting some $a_{\mathbf{k}}^{(1)}$, $a_{\mathbf{k}}^{(2)}$, $b_{\mathbf{k}}^{(+)}$ on $\langle 0|$.

Suppose now we take $\langle\psi'|\psi\rangle$, where $|\psi\rangle$ has at least one instance of $b_{\mathbf{k}}^{(+)}$. Since there are no instances of $b_{\mathbf{k}}^{(-)\dagger}$ operators, the $b_{\mathbf{k}}^{(+)}$ commutes with all the raising/lowering operators in the product. One can thus commute it all the way to the left, where it annihilates $\langle 0|$. Therefore, $\langle\psi'|\psi\rangle = 0$. This includes the case where $|\psi'\rangle = |\psi\rangle$, so in particular a state with a $b_{\mathbf{k}}^{(+)\dagger}$ excitation has zero norm.

The same proof holds if $|\psi\rangle$ is a linear combination of states, each given by acting creation operators on the vacuum, where at least one has a $b_{\mathbf{k}}^{(+)\dagger}$ excitation.

(g) Show that any state in $\mathcal{H}_{\text{small}}$ with non-zero norm must have the form

$$|\psi\rangle = |\psi_T\rangle + |\chi\rangle$$

where $|\psi_T\rangle$ contains only excitations of $a_{\mathbf{k}}^{(1)\dagger}$, $a_{\mathbf{k}}^{(2)\dagger}$ (i.e. transverse components), and $|\chi\rangle$ a null state. $|\psi\rangle$ should be physically equivalent to $|\psi_T\rangle$, as they differ only by a null state. We can then forget about the null states, and define

$$\mathcal{H}_{\text{phys}} = \{|\psi_T\rangle\}$$

$\mathcal{H}_{\text{phys}}$ contains only positive-norm states, and is identical to that obtained in the Coulomb gauge.
We showed in (f) that any state with non-zero $b_{\mathbf{k}}^{(+)\dagger}$ excitations is a null state. Since we can expand an arbitrary state as a sum of n-particle states using creation and annihilation operators, we can write any state in $|\psi\rangle \in \mathcal{H}_{\text{small}}$ as

$$|\psi\rangle = |\psi_T\rangle + |\chi\rangle$$

where we put all terms with non-zero $b_{\mathbf{k}}^{(+)\dagger}$ excitations into $|\chi\rangle$, and all terms without $b_{\mathbf{k}}^{(+)\dagger}$ excitations into $|\psi_T\rangle$. By definition, $|\chi\rangle$ is a null state, and $|\psi_T\rangle$ contains only $a_{\mathbf{k}}^{(1)\dagger}$, $a_{\mathbf{k}}^{(2)\dagger}$ excitations.

In particular, one finds $\langle\psi|\psi\rangle = \langle\psi_T|\psi_T\rangle > 0$, which proves (b) in full generality.

(h) Let us call excitations of $b_{\mathbf{k}}^{(+)\dagger}$ null photons. To understand their physical interpretation, consider the 'wave-function' $\chi_\mu(x)$ if the single null photon state

$$\chi_\mu(x) = \langle 0|A_\mu(x)|\mathbf{k},+\rangle, \qquad |\mathbf{k},+\rangle = \sqrt{2\omega_{\mathbf{k}}}\,b_{\mathbf{k}}^{(+)\dagger}|0\rangle$$

Note that the above definition of wavefunction $\chi_\mu(x)$ is the straightforward generalization to vector fields of our previous discussion for scalar fields.
Show that $\chi_\mu(x)$ can be written as

$$\chi_\mu(x) = \partial_\mu\lambda(x)$$

where $\lambda(x)$ is some function which satisfies the equation for a massless scalar

$$\partial_\mu\partial^\mu\lambda = 0$$

This shows that a null photon can be interpreted as a guage transformation from the vacuum. To see this, recall that the Lorentz gauge $\partial_\mu A_\mu = 0$ leaves residual gauge transformations

$$A_\mu \to A_\mu + \partial_\mu\lambda, \qquad \partial_\mu\partial^\mu\lambda = 0$$

Thus, $\chi_\mu(x)$ can be considered a residual gauge transformation from $A_\mu = 0$.
Using the mode expansion for $A_\mu(x)$, we have

$$\begin{aligned}\chi_\mu(x) = \langle 0|A_\mu(x)|\mathbf{k},+\rangle &= \int d\!\!{}^-{}^3\mathbf{p}\sqrt{\frac{\omega_{\mathbf{k}}}{\omega_{\mathbf{p}}}}e^{ip\cdot x}\epsilon_\mu^{(\alpha)}\langle 0|a_{\mathbf{p}}^{(\alpha)}b_{\mathbf{k}}^{(+)\dagger}|0\rangle \\ &= \frac{1}{\sqrt{2}}e^{ik\cdot x}(-\epsilon_\mu^{(0)}+\epsilon_\mu^{(3)}) = \frac{1}{\sqrt{2\omega_{\mathbf{k}}}}e^{ik\cdot x}k_\mu = -\frac{i}{\sqrt{2\omega_{\mathbf{k}}}}\partial_\mu(e^{ik\cdot x})\end{aligned}$$

In the second line we used the explicit forms of $\epsilon_\mu^{(\alpha)}$, and that $k_\mu = (-\omega_{\mathbf{k}}, \mathbf{k})$. Therefore, $\chi_\mu(x) = \partial_\mu\lambda(x)$ with $\lambda(x) = -\frac{i}{\sqrt{2\omega_{\mathbf{k}}}}e^{ik\cdot x}$. Since $k^2 = 0$, we immediately have $\partial^2\lambda = 0$.

(i) Show that the conclusion of part (h) holds for any wavepacket of a null photon,

$$|f\rangle = \int \mathrm{d}^3\mathbf{k}\, f(\mathbf{k})|\mathbf{k},+\rangle$$

More generally, for $|f\rangle = \int d^3\mathbf{k} f(\mathbf{k})|\mathbf{k},+\rangle$, we compute

$$\langle 0|A_\mu(x)|\mathbf{k},+\rangle = \int \frac{d^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}e^{ik\cdot x}f(\mathbf{k})k_\mu = \partial_\mu\tilde{f}(x)$$

where $\tilde{f}(x) := -i\int \frac{d^3\mathbf{k}}{\sqrt{2\omega_{\mathbf{k}}}}f(\mathbf{k})e^{ik\cdot x}$. We again have $k_\mu = (-\omega_{\mathbf{k}}, \mathbf{k})$ and $k^2 = 0$, so all the results from (h) carry over.

Question 2: Casimir Effect in 1 Dimension (30 points)

Until now, we have disregarded the vacuum's zero-point energy as an unobservable (infinite) shift in the energy scale. However, as Casimir demonstrated in 1948, differences in vacuum zero point energies are observable. This phenomenon is known as the Casimir effect. A simplest example is a small attractive force between 2 close parallel conducting plates, due to quantum vacuum fluctuations of the EM field. The force is caused by a change in vacuum energy of the EM field which results from the boundary conditions imposed by the plates.

In this problem, we will explore the Casimir effect. For technical simplicity, we consider a toy version. As we have seen, after quantization the EM field has the same number of degrees of freedom as 2 massless scalar fields. Thus, consider a free, massless, real scalar field ϕ in 1 spatial dimension,

$$S = -\frac{1}{2}\int \mathrm{d}x\mathrm{d}t\partial_\mu\phi\partial^\mu\phi$$

The vacuum of the system has infinite zero point energy. Denote it by E_0. Now imagine we put 2 'plates' at $x = 0$ and $x = a$ such that ϕ is required to vanish at the location of the plates,

$$\phi(x = 0, t) = \phi(x = a, t) = 0$$

Adding plates which impose additional boundary conditions on ϕ disturbs the vacuum, and results in a different zero-point energy $E(a)$.

Even though E_0 and $E(a)$ are both infinite, their different turns out to be finite, and physically meaningful. In fact, the difference

$$U(a) = E(a) - E_0$$

can be considered as the potential energy between the 2 plates. Changing a modifies the potential energy, resulting in a force between the plates which can be measured experimentally.

In this problem we compute $U(a)$. Taking the difference between infinities is a highly dangerous thing to do, and in principle one can get any answer, so we will need to be careful.

Both E_0 and $E(a)$ have 2 sources of infinities, one from the infinite volume, the other from there being infinite local degrees of freedom. It is convenient to separate them by putting the system in a box with finite size $L \gg a$. We will take $L \to \infty$ at the end. More explicitly, we require ϕ to satisfy a periodic boundary condition corresponding to a circle of size L,

$$\phi(x, t) = \phi(x + L, t)$$

(a) In the vacuum (i.e. before putting plates), write down the mode expansion for ϕ and calculate its zero-point energy E_0. Your answer should have the form

$$E_0 = \frac{1}{2}\sum_n \omega_n$$

where ω_n is the energy for each mode. Specify both ω_n and the range of summation.

We are interested in a real scalar field with periodicity L, i.e. $\phi(x + L, t) = \phi(x, t)$. We write ϕ in terms of creation and annihilation operators, this time restricting to modes apropriate for the circle.

$$\phi(x, t) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{2L\omega_n}}(a_n e^{-i\omega_n t + ik_n x} + a_n^\dagger e^{i\omega_n t - ik_n x}), \qquad k_n = \frac{2\pi n}{L}, \quad \omega_n = \frac{2\pi|n|}{L}$$

Substituting this expansion into the Hamiltonian, and using the commutator $[a_m, a_n^\dagger] = \delta_{m,n}$, one has

$$H = \sum_{n=-\infty}^{\infty} \omega_n N_n + \frac{1}{2} \sum_{n=\infty}^{\infty} \omega_n$$

We find that

$$E_0 = \frac{1}{2} \sum_{n\in\mathbb{Z}} \frac{2\pi|n|}{L} = \frac{1}{2} \sum_{n\geq 1} \frac{4\pi n}{L}$$

(b) Adding the plates separates the system into 2 segments, one with size a, and one with size $L-a$. In both segments one has Dirichlet boundary conditions at either end. Therefore it is enough to work out one of them. Find the mode expansion for ϕ in the region $[0, a]$ and zero point $\varepsilon(a)$. Again your answer should have the form

$$\varepsilon(a) = \frac{1}{2} \sum_n \tilde{\omega}_n$$

Specify both $\tilde{\omega}_n$ and the range of summation. The total zero-point energy of the system in the presence of the plates is thus

$$E(a) = \varepsilon(a) + \varepsilon(L-a)$$

We proceed as before, but now write ϕ in terms of modes appropriate for Dirichlet boundary conditions on the interval $[0, a]$:

$$\phi(x,t) = \sum_{n=1}^{\infty} \frac{\sin \tilde{\omega} x}{\sqrt{a\tilde{\omega}_n}} (a_n e^{-i\tilde{\omega}_n t} + a_n^\dagger e^{i\tilde{\omega}_n t}), \qquad \tilde{\omega}_n = \frac{\pi n}{a}$$

Substituting this expansion into the Hamiltonian, and using the commutator $[a_m, a_n^\dagger] = \delta_{m,n}$, one has

$$H = \sum_{n=1}^{\infty} \tilde{\omega}_n N_n + \frac{1}{2} \sum_{n=\infty}^{\infty} \tilde{\omega}_n$$

We find that

$$\varepsilon(a) = \frac{1}{2} \sum_{n\geq 1} \frac{\pi n}{a}$$

(c) Both sums in (a) and (b) are divergent. There is not much sense in taking their difference. To do this, we will first make them finite by introducing a 'UV-cutoff' Λ, and change the sums to

$$E_0 = \frac{1}{2} \sum_n \omega_n e^{-\omega_n/\Lambda}, \qquad \epsilon(a) = \frac{1}{2} \sum_n \tilde{\omega}_n e^{-\tilde{\omega}_n/\Lambda}$$

Evaluate these expressions with finite Λ.

Using the identity

$$\sum_{n=1}^{\infty} n e^{-an} = \frac{e^a}{(e^a-1)^2} = \frac{1}{4\sinh^2(a/2)}$$

We find

$$E_0 = \frac{1}{2}\sum_{n\geq 1} \frac{4\pi n}{L} e^{-\frac{2\pi n}{L\Lambda}} = \frac{\pi}{2L\sinh^2(\pi/L\Lambda)}$$

$$\varepsilon(a) = \frac{1}{2}\sum_{n\geq 1} \frac{\pi n}{a} e^{-\frac{\pi n}{a\Lambda}} = \frac{\pi}{8a\sinh^2(\pi/2a\Lambda)}$$

(d) Expand the answers from part (c) in the limit $\Lambda \to \infty$. You will find they become divergent. Keep terms which are divergent and finite, but throw out all terms which go to zero in the limit.

We obtain

$$E_0 = \frac{\pi}{2L\sinh^2(\pi/L\Lambda)} \sim \frac{L\Lambda^2}{2\pi} - \frac{\pi}{6L} + \mathcal{O}(\Lambda^{-2})$$

$$\varepsilon(a) = \frac{\pi}{8a\sinh^2(\pi/2a\Lambda)} \sim \frac{a\Lambda^2}{2\pi} = \frac{\pi}{24a} + \mathcal{O}(\Lambda^{-2})$$

(e) From your answers in part (d), find

$$U(a) = E(a) - E_0$$

You should find $U(a)$ is finite in the limit $\Lambda \to \infty$. Now take the limit $L \to \infty$ in $U(a)$, and find the force between the plates.

Combining the above terms, we have

$$U(a) = \varepsilon(a) + \varepsilon(L-a) - E_0 = \frac{\pi}{6L} - \frac{\pi}{24a} - \frac{\pi}{24(L-a)}$$

Taking $L \to \infty$ gives

$$U(a) = -\frac{\pi}{24a}, \qquad F(a) = -\partial_a U(a) = -\frac{\pi}{24a^2}$$

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