Recitation 5

Wentao Cui

March 17, 2023

1 Path Integrals in QFT

1.1 Bridge: to Infinite Degrees of Freedom

As we've discussed, the path-integral formulation of QFT is tantamount to Gaussian integrals with a infinite number of degrees of freedom. It is thus important to discuss how calculus and linear algebra change when we go from a finite number of degrees of freedom to an infinite number, often in a continuous domain. In this section we establish a dictionary between them.

Degrees of freedom:

- Finite vector $\to$ continuous function, i.e. $\phi_i \to \phi(x)$
  When working with Gaussians, we had degrees of freedom ϕ_i. For i continuous, like a space or a time, this becomes a function $\phi(x)$. For our QFT purposes this will be a field depending on space and time. Often we might want both continuous and discrete labels, such as a vector field $A_i(x)$. One can also view QM as a $0+1$-dimensional QFT. Here the degree of freedom is time, and the 'fields' are the positions X_i.

Basic operations:

- Sums $\to$ integrals, i.e. $\sum_i \to \int dx$
- Functions $\to$ functionals, i.e. $f(\phi_i) \to F[\phi(x)]$
  The functionals of our interest include things like the action or Lagrangian. Generally they will be an integral of some function of $\phi(x)$, and can also be functions of other points x':

$$F[\phi(x)] = \int dx f(\phi(x)), \qquad H_{x'}[\phi(x)] = \int_{\mathbb{R}} dx h(x, x')\phi(x)$$

Calculus:

- (Partial) derivatives $\to$ functional derivatives, i.e. $\dfrac{\partial}{\partial \phi_i} \to \dfrac{\delta}{\delta \phi(x)}$
  A normal derivative measures how much a function changes with respect to a finite degree of freedom. A functional derivative measures how much a functional changes with respect to a function (infinite degree of freedom). This can be made explicit for some $F[\phi(x)]$ by perturbing by some test function $\eta(x)$, and seeing how much F varies:

$$\Delta F[\phi(x), \eta(x)] = \lim_{\epsilon \to 0} \frac{F[\phi(x) + \epsilon\eta(x)] - F[\phi(x)]}{\epsilon} = \int dx \eta(x) \frac{\delta F}{\delta \phi(x)}$$

  It is instructive to compare this to a directional derivative, which we have for discrete ϕ_i, where we have some vector-perturbation n_i.

$$\Delta f(x_i, n_i) = \lim_{\epsilon \to 0} \frac{f(x_i + \epsilon n_i) - f(x_i)}{\epsilon} = \sum_i n_i \frac{\partial f}{\partial x_i}$$

We recover the partial derivative in the j-direction for $n_i = \delta_{ij}$, and similarly we recover the functional derivative in the x_0-th direction for $\eta(x) = \delta(x - x_0)$. This measures the change in F when we add a small perturbation at x_0. [DRAW FIGURE]

- Integrals $\rightarrow$ path integrals, i.e. $\int \prod_i d\phi_i \rightarrow \int D\phi(x)$
  This is more intuitive. One has infinite degrees of freedom, so we have to integrate over each. One can think of integrating over the values of a field each each point in spacetime.

Linear algebra:

- Matrices $\rightarrow$ operators, i.e. $A_{ij} \rightarrow K(x, x')$
  This should be familiar from quantum mechanics. A matrix is just a finite dimensional operator. In the continuous case the vector space on which the operator acts is infinite-dimensional and each index becomes a continuous variable.
  The identity operator $\mathbb{1}_{ij}$ becomes the Dirac-delta, $\delta(x-x')$. We can also have differential operators like ∂_x, which don't have a relevant finite analog (it would look something like $\Delta\phi_i = \phi_i - \phi_{i-1}$, which generically does not have any meaning).

- Matrix products: $\rightarrow$ operator products, i.e. $A_{ij}B_{jk}\phi_k \rightarrow \int dx'dx'' A(x,x')B(x',x'')\phi(x'')$
  Index sums become integrals. We can confirm that the Dirac-delta is the identity, since

$$\int dx' A(x, x')\delta(x - x'') = A(x, x'')$$

- Matrix inverses $\rightarrow$ Green's functions, i.e. $A^{-1}_{ij} \rightarrow G_K(x, x')$
  The matrix inverse is defined as the operator that, when multiplied with the original operator, yields the identity:

$$\sum_j A_{ij} A^{-1}_{jk} = \delta_{ik}$$

  Using the above we take the continuous analog:

$$\int dx' K(x, x')K^{-1}(x', x'') = \delta(x - x'')$$

  We recognize this as the Green's function for the operator K. For instance for the Klein-Gordon operator we have

$$\int dx' \delta^{(4)}(x - x')(-\partial^2_{x'} + m^2)G(x', x'') = \delta^{(4)}(x - x'') \quad \Longleftrightarrow \quad (-\partial^2_x + m^2)G(x, x'') = \delta^{(4)}(x - x'')$$

- Eigendecompositions:
  These are defined in the same way

$$\sum_i A_{ij}v_j - \lambda v_j \rightarrow \int dx' K(x, x')f(x') = \lambda f(x)$$

  While the eigenvalues stay numbers, eigenvectors now become functions. This should be familiar from quantum mechanics: the eigenvectors of the derivative operator $K(x, x') = \delta(x - x')\frac{d}{dx}$ are just the exponential functions $e^{\lambda x}$ with eigenvalues $\lambda \in \mathbb{R}$:

$$\int dx' \delta(x - x')\frac{d}{dx}e^{\lambda x'} = \frac{d}{dx}e^{\lambda x} = \lambda e^{\lambda x}$$

- Determinants and traces:
  These are defined as usual: as the sum/product of the operator's eigenvalues. In the continuous case, we will have an infinite sum (i.e. integral) or product. Often these will be infinite, but taking ratios of determinants or traces of operators gives something finite.

1.2 Path-Integrals for Free Theories

Note: for this section I would recommend opening the notes from the previous recitation and compare this to what I'll be doing today. You will see that the procedure is almost identical.

1.2.1 The Distribution

All the hard work is behind us. We can use the same techniques we had for computing correlators of finite Gaussian integrals to path-integrals. In particular, the Wick contractions we saw last week will give rise to Feynman diagrams.

The path-integral for a field theory in n dimensions is based on the following distribution:

$$p[\phi(x)] = \frac{1}{Z_0} e^{iS[\phi]}, \qquad \phi \in \mathbb{R}^{1,n-1}$$

where we define the partition function Z_0, for now thought of as a normalization constant

$$\mathcal{Z}_0 = \int D[\phi(x)] e^{iS[\phi(x)]}$$

By free field theory, we mean that the action is quadratic:

$$\begin{aligned}
iS &= \int d^4x \left[\frac{i}{2}(-\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)\right] = \frac{1}{2}\int d^4x \phi(\partial^2 - m^2)\phi \\
&= \frac{i}{2}\int d^4x d^4y\, \phi(y)\left[\delta^{(4)}(x-y)(\partial_x^2 - m^2)\right]\phi(x) \\
&= \frac{i}{2}\int đ^4p đ^4q\, \phi(q)\left[(2\pi)^4\delta^{(4)}(p-q)(-p^2-m^2)\right]\phi(p) \\
&= -\frac{1}{2}\int đ^4p đ^4q\, \phi(q)K(q,p)\phi(p), \qquad\qquad K(q,p) = i(2\pi)^4\delta^{(4)}(p-q)(p^2+m^2)
\end{aligned}$$

In the last line I write this in momentum space. This looks almost identical to a finite Gaussian integral, where instead of $\phi_i A_{ij}\phi_j$ we have $\phi(p)K(p,q)\phi(q)$. In momentum space K is particularly simple, with no derivatives. Note that the $\delta^{(4)}(p-q)$ makes this operator diagonal: we have no mixing between $\phi(p)$ and $\phi(q)$ unless $p = q$. If you made me write this as a finite dimensional operator, it would morally be

$$A_{jk} = i(k^2+m^2)\delta_{jk} = i\,\mathrm{diag}(\cdots, 4+m^2, 1+m^2, m^2, 1+m^2, 4+m^2, \cdots)$$

This way of viewing makes the Green's function, or matrix inverse, particularly clear. Proceeding with the matrix analogy, the inverse is

$$A^{-1}_{jk} = \frac{-i}{k^2+m^2}\delta_{jk} = \mathrm{diag}\left(\cdots, \frac{-i}{4+m^2}, \frac{-i}{1+m^2}, \frac{-i}{m^2}, \frac{-i}{1+m^2}, \frac{-i}{4+m^2}, \cdots\right)$$

We thus can immediately write down the Green's function,

$$G(q,p) = \delta(p-q)\frac{-i}{p^2+m^2}$$

1.2.2 Correlators

We seek to compute correlation functions of operators with respect to this 'probability distribution':

$$G_n := \langle 0|\mathrm{T}\hat{\phi}(x_n)\cdots\hat{\phi}(x_1)|0\rangle = \frac{1}{Z_0}\int D\phi e^{iS[\phi]}\phi(x_n)\cdots\phi(x_1)$$

Note that on the left, $\hat{\phi}$'s are operators acting on a Hilbert space, while on the right the ϕ's are functions. These correlators give us information about fluctuations of the field around the vacuum state.

To compute these objects we introduce the generating functional

$$\begin{aligned} Z[J] &= \int D\phi \, \exp\left[\frac{i}{2}\int \bar{d}^4p\bar{d}^4q\, \phi(q)K(q,p)\phi(p) + i\int \bar{d}^4p\, J(p)\phi(p)\right] \\ &= \frac{C}{\sqrt{\det K}} \exp\left[\frac{i}{2}\int \bar{d}^4p\bar{d}^4qJ(q)K^{-1}(q,p)J(p)\right] \end{aligned}$$

In the last line we have completed the square to perform the functional integral. This technique is almost identical to what we did for matrices. The factor of C is the product of all the 2π factors which follow from the Gaussian integral.

From last week's discussion we can immediately write down the formula for computing correlators:

$$\begin{aligned} \langle\phi(p_n)\cdots\phi(p_1)\rangle &= \frac{1}{Z(0)}\left(-i\frac{\partial}{\partial J(p_n)}\right)\cdots\left(-i\frac{\partial}{\partial J(p_1)}\right)_{J=0} Z(J) \\ &= \sum_{\text{Wick}} K^{-1}(p_a,p_b)\cdots K^{-1}(p_c,p_d) \end{aligned}$$

In the first line we have obligatory factors of $-i$ for each derivative due to our convention to have an i in front of $J(p)$ in the integral. In the last line, we sum over all Wick contractions of $\{p_1, p_2, \ldots, p_n\}$, best thought of as indices in parallel with the matrix case. If at any point you get confused about this, I suggest rephrasing your question in terms of Gaussian integrals. Note that here I work in momentum space, where K^{-1} is particularly simple, acting without derivatives. This is what is conventionally done in QFT. If you insisted I work in position space, the same process goes through, the only difference being the form of $K^{-1}(y,x)$ compared to $K^{-1}(q,p)$. To go back to position space, you can also just take the Fourier transform of the correlation function with respect to each of the momenta.

Most important is the 2-point function, for which there is a single Wick contraction to consider. We collect all our results:

$$\begin{aligned} K^{-1}(q,p) &= G_F(q,p) = \langle 0|\mathrm{T}\phi(q)\phi(p)|0\rangle = \frac{-i}{p^2+m^2-i\epsilon}\delta(p-q) \\ K^{-1}(x,y) &= G_F(x,y) = \langle 0|\mathrm{T}\phi(x)\phi(y)|0\rangle = \int \bar{d}^4k\frac{-i}{k^2+m^2-i\epsilon}e^{ik\cdot(x-y)} \end{aligned}$$

That is, the inverse of the kinetic term, the Feynman Green's function, and the 2-point correlator are all identical. We call this object the propagator.

Example: let us compute the 4-point function of a free scalar theory. We can copy the Wick contractions almost verbatim from last week's example:

$$\begin{aligned} \langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle &= \langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle + \langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle + \langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \\ &= K^{-1}(x_1,x_2)K^{-1}(x_3,x_4) + K^{-1}(x_1,x_3)K^{-1}(x_2,x_4) + K^{-1}(x_1,x_4)K^{-1}(x_2,x_3) \end{aligned}$$

[DRAW DIAGRAMMATICALLY]

2 Interactions and The Feynman Calculus

2.1 Adding Interactions

Everything we have done is for a free theory, where we have only a quadratic term in our field ϕ. Free theories lead to Gaussian path-integrals, which is one of the very few cases where we can perform the path-integral explicitly. We saw this above. Adding interactions will lead to 2 major difficulties:

- More complicated Lagrangians. Generically, an action will have cubic and higher-order terms, such as a ϕ^4 theory:

  $$\mathcal{L} = -\partial_\mu\phi\partial^\mu\phi - m^2\phi^2 + \lambda\phi^4$$

  For generic cases like this, we can only compute the path-integral perturbatively, expanding around a free theory assuming small λ. The convergence of this series is a tricky business.

- A different vacuum. We denote the vacuum of a free theory by $|0\rangle$, and the interacting theory by $|\Omega\rangle$. These are different states: adding terms to the Lagrangian changes the vacuum. Since free theories are exactly solvable, we understand $|0\rangle$ much better than $|\Omega\rangle$. Perturbation theory will allow us to understand fluctuations about $|\Omega\rangle$ in terms of those about $|0\rangle$.

Now we turn to the evaluation of the correlator of an interacting theory:

$$G_n := \langle\Omega|\mathrm{T}\phi(x_1)\cdots\phi(x_n)|\Omega\rangle$$

We will do this in perturbation theory, in principle to an arbitrary order. Again the key ingredient will be Wick's theorem. Nothing about what we do is unique to field theories: the same process holds for matrix theories, if we add higher order terms to our Gaussian integrals.

To do this, we split our action into a free and interaction piece:

$$S = S_0 + S_I, \qquad\qquad S_0 = \int d^4x \left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2\right)$$

From class, we have the central 'magic formula' of Gell-Mann and Low which relates correlators about the interaction vacuum to those about the free vacuum:

$$G_n = \frac{\langle 0|\mathrm{T}\phi(x_1)\cdots\phi(x_n)e^{iS_I}|0\rangle}{\langle 0|\mathrm{T}e^{iS_I}|0\rangle}, \qquad\qquad S_I = -\int dt H_I$$

Perturbation theory, then, is the process of expanding the e^{iS_I} exponential to a given order and computing correlators. By now, you know 2 methods to compute these correlators, which give us 2 ways of doing perturbation theory. They amount to doing the same thing, which are Wick contractions.

1. Hamiltonian framework: using the free-field expansions of ϕ into creation and annihilation operators

2. Path-integral framework: view these as correlators of a Gaussian density, and perform the Gaussian integrals explicitly.

This formula contains all the physics, and is something all of you should remember. The rest is just combinatorics, and will be handled in the next recitation.

MIT OpenCourseWare
https://ocw.mit.edu

8.323 Relativistic Quantum Field Theory I
Spring 2023

For information about citing these materials or our Terms of Use, visit: https://ocw.mit.edu/terms.