Problem Set 3 Solutions

1. (a) Let's denote the Lorentz transformation of p as $\tilde{p} = \Lambda p$. Since $p = 0 \Leftrightarrow \tilde{p} = 0$ this implies that $\delta^4(p) = C\delta^4(\tilde{p})$ for some constant C. Then, for some function of momentum f,

$$f(0) = \int d^4\tilde{p}\, f(\tilde{p})\, \delta^4(\tilde{p}) = \int d^4p\, |\det\Lambda| f(\Lambda p)\; C\delta^4(p) = Cf(0) \tag{1}$$

implying $C = 1$ and $\delta^4(p) = \delta^4(\tilde{p})$.

(b) Note that assuming $p^0 = \omega_{\vec{p}}$

$$\begin{aligned}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}} f(p) &= \int \frac{d^4p}{(2\pi)^3}\delta\left(p^2+m^2\right)\theta\left(p^0\right) f(p) \\ &= \int \frac{d^4\widetilde{p}}{(2\pi)^3}\delta\left(\widetilde{p}^2+m^2\right)\theta\left(\widetilde{p}^0\right) f(\widetilde{p}) \\ &= \int \frac{d^4p}{(2\pi)^3}\delta\left(p^2+m^2\right)\theta\left(p^0\right) f(\widetilde{p}) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{2\omega_{\vec{p}}} f(\widetilde{p})\end{aligned} \tag{2}$$

In the second line we renamed $p \to \widetilde{p}$, in the third we used $|\det\Lambda| = 1$, $\widetilde{p}^2 = p^2$ and $\theta\left(\widetilde{p}^0\right) = \theta\left(p^0\right)$ for Lorentz transformations connected to the identity. Then the identity we aimed at follows.

(c) Let's examine the matrix element

$$\langle 0|\phi(0)|k\rangle \tag{3}$$

Let U be the unitary operator on the Hilbert space that implements a boost from $\vec{k}$ back to the rest frame of the particle at zero momentum. Then

$$\langle 0|\phi(0)|k\rangle = \langle 0|U^{-1}U\phi(0)\,U^{-1}U|k\rangle = \langle 0|\phi(0)\,U|k\rangle = \langle 0|\phi(0)|m\rangle \tag{4}$$

where $|m\rangle$ is the one particle state at zero momentum. Above, we have used the Lorentz invariance of the vacuum, and the Lorentz invariance of a scalar field operator (the result would have been different for spinor or vector fields). Thus this matrix element is actually k-independent, and so is its norm

$$Z = |\langle 0|\phi(0)|k\rangle|^2 \tag{5}$$

2. (a) The relevant transition amplitude is that between a state of one ϕ-particle at time minus infinity and a state with two χ particles at time plus infinity:

$$\langle k_1, k_2; +\infty|p_\phi; -\infty\rangle = \langle k_1 k_2|T\exp\left(i\int d^4x\; \frac{g}{2}\phi\chi^2\right)|p_\phi\rangle \tag{6}$$

$$= i\frac{g}{2}\langle k_1k_2|\int d^4x\;\phi\chi^2|p_\phi\rangle + O\left(g^2\right)\,. \tag{7}$$

$$= i\frac{g}{2}\int d^4x\; e^{i(-k_1-k_2+p)\cdot x}\langle k_1k_2|\phi(0)\,\chi(0)^2|p_\phi\rangle \tag{8}$$

$$= i\frac{g}{2}\,(2\pi)^4\,\delta^4(k_1+k_2-p)\cdot 2\,, \tag{9}$$

where the factor of two comes from the two possible pairings with the external particle. We thus read $T(k_1, k_2; p) = -g$. The decay rate is then given by

$$\Gamma = \frac{1}{2}\frac{1}{(2\pi)^2}\int \frac{d^3k_1}{2E(k_1)}\frac{d^3k_2}{2E(k_2)}\delta^{(4)}(k_1+k_2-p)\,\frac{g^2}{2p^0}\,. \tag{10}$$

The extra factor of (1/2) in front of the integral is there because the two outgoing particles are indistinguishable and we must not overcount final states. Writing

$$\Gamma = \frac{g^2}{16\pi^2}\frac{1}{M}\int \frac{d^3k_1}{2E(k_1)}\frac{d^3k_2}{2E(k_2)}\delta^{(4)}(k_1+k_2-p)\ , \tag{11}$$

We choose a Lorentz frame where $p^\mu = \left(M,\vec{0}\right)$. The decay rate becomes

$$\Gamma = \frac{g^2}{16\pi^2}\frac{1}{M}\int \frac{d^3\vec{k}}{2E\left(\vec{k}\right)}\int \frac{d^3\vec{k'}}{2E'\left(\vec{k'}\right)}\delta\left(E\left(\vec{k}\right)+E\left(\vec{k'}\right)-M\right)\delta^{(3)}\left(\vec{k}+\vec{k'}\right)\ . \tag{12}$$

Integrating over $\vec{k'}$ we get

$$\Gamma = \frac{g^2}{16\pi^2}\frac{1}{M}\int \frac{d^3\vec{k}}{4\left(E\left(\vec{k}\right)\right)^2}\ \delta\left(2E\left(\vec{k}\right)-M\right)\ . \tag{13}$$

The value $\bar{k}$ of $|\vec{k}|$ that solves the energy conservation equation $M = 2E\left(\vec{k}\right) = 2\sqrt{m^2+\vec{k}^2}$ is

$$\bar{k} = \sqrt{\frac{M^2}{4}-m^2} \tag{14}$$

Thus, changing variables in the delta function

$$\Gamma = \frac{g^2}{16\pi^2}\frac{1}{M}\int_0^\infty \frac{\pi k^2 dk}{E(k)^2}\ \frac{\delta\left(k-\bar{k}\right)}{\frac{2k}{E(k)}} = \frac{g^2}{16\pi^2}\frac{1}{M}\left(\frac{\pi\bar{k}}{M}\right) = \frac{g^2}{16\pi}\frac{\bar{k}}{M^2} = \frac{g^2}{32\pi M}\sqrt{1-\left(\frac{2m}{M}\right)^2} \tag{15}$$

(b) This time the in/out matrix element is

$$\begin{aligned}\langle k_B, k_C, k_D; +\infty | k_A; -\infty\rangle &= -ig\int dx\langle k_B|B(x)|0\rangle\langle k_C|C(x)|0\rangle\langle k_D|D(x)|0\rangle\langle 0|A(x)|k_A\rangle \\ &= -ig\ (2\pi)^4\ \delta^{(4)}(k_B+k_C+k_D-k_A)\ .\end{aligned} \tag{16}$$

We thus read $T(k_B, k_C, k_D; k_A) = g$. Since the particles B, C, and D are massless, their energies are equal to the magnitude of their momenta (say, $k_B^0 = |\vec{k}_B|$). Thus

$$\begin{aligned}\Gamma &= \frac{1}{(2\pi)^5}\int \frac{d^3\vec{k}_B}{2|\vec{k}_B|}\frac{d^3\vec{k}_C}{2|\vec{k}_C|}\frac{d^3\vec{k}_D}{2|\vec{k}_D|}\delta^{(4)}(k_B+k_C+k_D-k_A)\frac{g^2}{2m} \\ &= \frac{g^2}{(2\pi)^5}\frac{1}{16m}\int \frac{d^3\vec{k}_B}{|\vec{k}_B|}\frac{d^3\vec{k}_C}{|\vec{k}_C|}\frac{d^3\vec{k}_D}{|\vec{k}_D|}\ \delta^{(3)}\left(\vec{k}_B+\vec{k}_C+\vec{k}_D\right)\delta\left(|\vec{k}_B|+|\vec{k}_C|+|\vec{k}_D|-m\right) \\ &= \frac{g^2}{(2\pi)^5}\frac{1}{16m}\int \frac{d^3\vec{k}_B}{k_B}\frac{d^3\vec{k}_C}{k_C}\frac{d^3\vec{k}_D}{k_D}\ \delta^{(3)}\left(\vec{k}_B+\vec{k}_C+\vec{k}_D\right)\delta\left(k_B+k_C+k_D-m\right)\end{aligned} \tag{17}$$

where, in the last step, we defined $k = |\vec{k}|$ for particles B, C, and D. We notice that the vector $\vec{k}_D$ is fully determined once we fix $\vec{k}_B$ and $\vec{k}_C$. Doing the integral over $\vec{k}_D$,

$$\Gamma = \frac{g^2}{(2\pi)^5}\frac{1}{16m}\int \frac{d^3\vec{k}_B}{k_B}\frac{d^3\vec{k}_C}{k_C}\frac{1}{k_D}\delta\left(k_B+k_C+k_D-m\right)\ ,\quad \text{with}\quad k_D = |\vec{k}_B+\vec{k}_C|\ . \tag{18}$$

By rotational invariance we can imagine doing the $\vec{k}_C$ integral before the $\vec{k}_B$ integral and orienting $\vec{k}_B$ along the z axis. This integral will eliminate all angular dependence, and then we can take $d^3\vec{k}_B = 4\pi k_B^2 dk_B$. Thus

$$d^3\vec{k}_B\, d^3\vec{k}_C = 4\pi k_B^2 dk_B\ k_C^2 dk_C\ 2\pi\sin\theta d\theta\ . \tag{19}$$

We now trade the variable of integration θ for the variable of integration k_D. Note the ranges

$$\theta \in [0,\pi] \quad \rightarrow \quad |k_B - k_C| \leq k_D \leq k_B + k_C\,. \tag{20}$$

Since we are only changing one of the variables of integration it suffices to use

$$dk_D = \left|\frac{dk_D}{d\theta}\right| d\theta\,. \tag{21}$$

Since $k_B^2 + k_C^2 + 2k_B k_C \cos\theta = k_D^2$ we obtain

$$dk_D = \frac{k_B k_C}{k_D}\sin\theta\, d\theta\,. \tag{22}$$

Back in (19)

$$d^3\vec{k}_B\, d^3\vec{k}_C = 8\pi^2\, k_B dk_B\, k_C dk_C\, k_D dk_D\,, \tag{23}$$

and the decay rate becomes

$$\Gamma = \frac{g^2}{64\pi^3 m}\int_{|k_B-k_C|\leq k_D\leq k_B+k_C} dk_B\, dk_C\, dk_D \delta\left(k_B + k_C + k_D - m\right)\,. \tag{24}$$

The region of integration is nicely represented using Cartesian k_B, k_C, and k_D axes. Since all k are positive, the delta function restricts the integration to the portion of the plane $k_B + k_C + k_D = m$ that lies in the first quadrant; an equilateral triangle with vertices on the axes, a distance m from the origin. The inequalities for k_D restrict further the domain to the "barycentric triangle" that is formed by joining the midpoints of the sides of the original triangle (see Figure 1). Because of the delta function, the value of the integral in (24) is given by the area of the projection of the barycentric triangle on any of the three planes, (k_B, k_C), for example. The projection is shown in the figure, and its area is $m^2/8$. We thus have

$$\Gamma = \frac{g^2 m}{512\pi^3}\,. \tag{25}$$

If $\mathcal{L}_{int} = -gAB^3$, particle A decays into three B particles. The contractions between the interaction and the final states give a combinatorial factor of 3!, so

$$T\left(k_B, k_C, k_D; k_A\right) = g \quad \rightarrow \quad T\left(k_{B_1}, k_{B_2}, k_{B_3}; k_A\right) = 6g\,. \tag{26}$$

Since the three decay particles are indistinguishable we must include a symmetry factor of $1/3! = 1/6$ in the phase space integral

$$\int dk_B dk_C dk_D \rightarrow \frac{1}{6}\int dk_{B_1} dk_{B_2} dk_{B_3} \tag{27}$$

Since T enters squared, we get $\Gamma_{new} = \frac{1}{6}\cdot 6^2\cdot\Gamma = 6\Gamma$.

3. We can perform the same sorts of manipulations that led us to the scalar Lehmann representation.

$$\langle 0|T\psi(x)\bar{\psi}(y)|0\rangle = \theta\left(x^0 - y^0\right)\langle 0|\psi(x)\bar{\psi}(y)|0\rangle - \theta\left(y^0 - x^0\right)\langle 0|\bar{\psi}(y)\psi(x)|0\rangle \tag{28}$$

We insert a complete set of states, finding,

$$\langle 0|\psi(x)\bar{\psi}(y)|0\rangle = \int\frac{d^3p}{(2\pi)^3}\sum_{s,\lambda}\frac{e^{-ip_\lambda\cdot(x-y)}}{2E_{p_\lambda}}\langle 0|\psi(0)|\lambda_p, s\rangle\langle\lambda_p, s|\bar{\psi}(0)|0\rangle \tag{29}$$

where s is the sum over all polarizations and λ is the sum over single and multi-particle states. We already made use of $\langle 0|\psi(0)|0\rangle = 0$. We observe that by the Wigner-Eckart theorem the sum only gives contributions

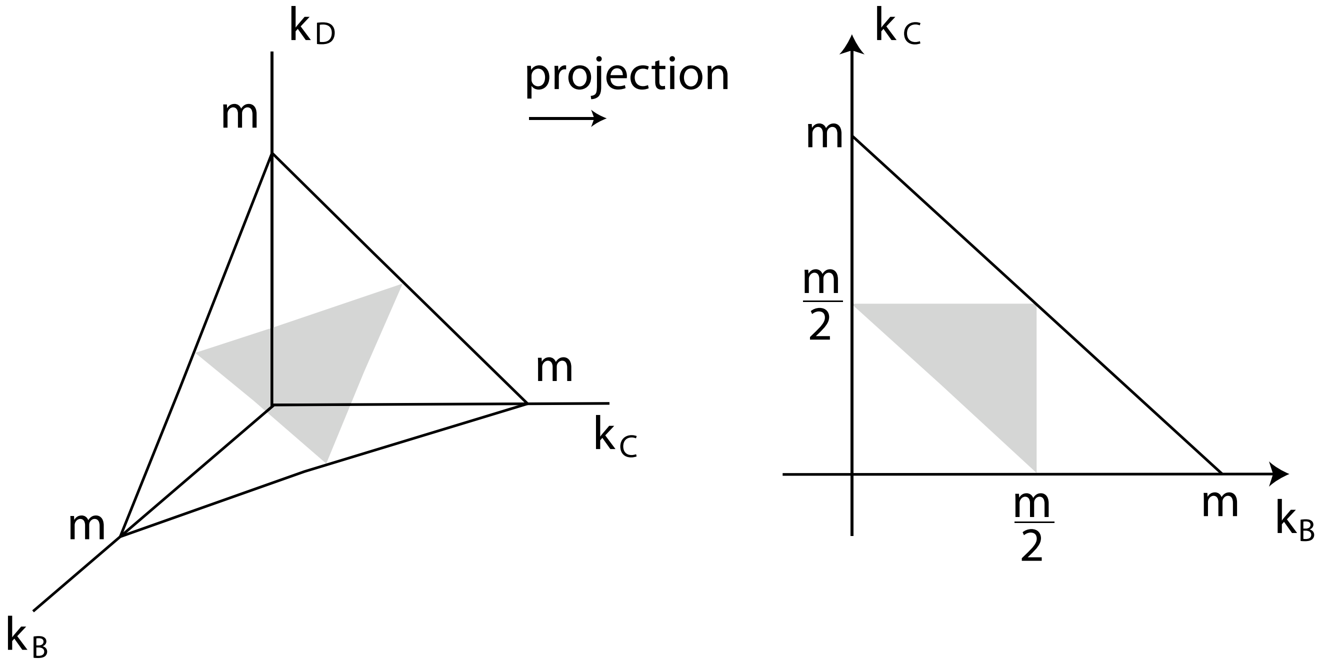

FIG. 1. The region of integration.

if $s=\pm\frac{1}{2}$. Performing the same manipulations that we used in class we get to:

$$\begin{aligned}\langle 0|\psi(x)\bar{\psi}(y)|0\rangle &= \int \frac{d^4p}{(2\pi)^4}\ e^{-ip\cdot(x-y)}\sum_{s,\lambda}2\pi\frac{\delta\left(p^0-p^0_\lambda\right)}{2E_p}\langle 0|\psi(0)|\lambda_p,s\rangle\langle\lambda_p,s|\bar{\psi}(0)|0\rangle \\ &= \int \frac{d^4p}{(2\pi)^4}\ e^{-ip\cdot(x-y)}\sum_{s,\lambda}2\pi\delta\left(p^2+m^2_\lambda\right)\langle 0|\psi(0)|\lambda_p,s\rangle\langle\lambda_p,s|\bar{\psi}(0)|0\rangle \\ &= \int \frac{d^4p}{(2\pi)^4}e^{-ip\cdot(x-y)}2\pi\Theta\left(p^0\right)\rho(p)\end{aligned} \tag{30}$$

$$2\pi\Theta\left(p^0\right)\rho(p) = \sum_{s,\lambda}2\pi\delta\left(p^2+m^2_\lambda\right)\langle 0|\psi(0)|\lambda_p,s\rangle\langle\lambda_p,s|\bar{\psi}(0)|0\rangle \tag{31}$$

Notice that we were *not* able to conclude that $\rho=\rho\left(-p^2\right)$ yet. As in the scalar case, we can insert operators that boost the ket back to $p=0$,

$$\langle 0|\psi(0)|p,s\rangle = \langle 0|U^\dagger U\psi(0)U^\dagger U|\lambda_p,s\rangle = \langle 0|S^{-1}\psi(0)|\lambda_0,s\rangle = S^{-1}\langle 0|\psi(0)|\lambda_0,s\rangle \tag{32}$$

where S belongs to the spin 1/2 representation of the Lorentz group and we have used the Lorentz invariance of the vacuum. For simplicity we will assume P, C, T invariance to conclude that:

$$\begin{aligned}\sum_s\langle 0|\psi(0)|\lambda_p,s\rangle\langle\lambda_p,s|\bar{\psi}(0)|0\rangle &= \sum_s S^{-1}\langle 0|\psi(0)|\lambda_0,s\rangle\langle\lambda_0,s|\psi(0)|0\rangle S = S^{-1}\left(D^{(\lambda)}_S+\left(D^{(\lambda)}_V\right)_\mu\gamma^\mu\right)S \\ &= D^{(\lambda)}_S+\left(D^{(\lambda)}_V\right)_\nu\Lambda^\nu{}_\mu\gamma^\mu\ ,\end{aligned} \tag{33}$$

where we used well known properties of Dirac matrices (e.g. PS (3.29)), the subscripts S and V denote scalar and vector respectively and we suppressed spinor indices. We used the assumed discrete symmetries to eliminate terms like $\left(D^{(\lambda)}_T\right)_{\mu\nu}\gamma^\mu\gamma^\nu$. Furthermore we only have one four vector quantity, the four velocity of the boost, so we can write:

$$\sum_s \langle 0|\psi(0)|\lambda_p, s\rangle\langle\lambda_p, s|\bar{\psi}(0)|0\rangle = D_S^{(\lambda)} + \left(D_V^{(\lambda)}\right)_\nu \Lambda^\nu{}_\mu\gamma^\mu = D_S^{(\lambda)} + \widetilde{D}_V^{(\lambda)}\not{p}_\lambda \tag{34}$$

$$2\pi\Theta\left(p^0\right)\rho(p) = \sum_\lambda 2\pi\delta\left(p^2 + m_\lambda^2\right)\left(D_S^{(\lambda)} + \widetilde{D}_V^{(\lambda)}\not{p}_\lambda\right) = 2\pi\Theta\left(p^0\right)\left[\rho_1\left(-p^2\right) + \rho_2\left(-p^2\right)\not{p}\,\right], \tag{35}$$

Because we know that there is only a single set of single particle states in our theory for that case we get:

$$\langle 0|\psi(0)|m, s\rangle = \sqrt{Z}u_s(0) \tag{36}$$

$$\sum_s \langle 0|\psi(0)|m, s\rangle\langle m, s|\bar{\psi}(0)|0\rangle = \sum_s Z\ S^{-1}\left(u_s(0)\,\bar{u}_s(0)\right)S = Z\ \sum_s u_s(p)\,\bar{u}_s(p) = Z\left(\not{p} - im\right)\ , \tag{37}$$

where we used the definition of single particle states. For $Z = 1$ we get back the free theory result. Now we are ready to determine the two point function.

$$\begin{aligned}
\langle 0|\psi(x)\,\bar{\psi}(y)|0\rangle &= \int \frac{d^4p}{(2\pi)^4}\ e^{-ip\cdot(x-y)}2\pi\Theta\left(p^0\right)\left[\rho_1\left(-p^2\right) + \rho_2\left(-p^2\right)\not{p}\ \right] \\
&= \int_0^\infty d\mu^2\ \int \frac{d^4p}{(2\pi)^4}\ e^{-ip\cdot(x-y)}2\pi\delta\left(p^2+\mu^2\right)\Theta\left(p^0\right)\left[\rho_1\left(\mu^2\right) + \rho_2\left(\mu^2\right)\not{p}\ \right] \\
&= \int_0^\infty d\mu^2\ \int \frac{d^4p}{(2\pi)^4}\ e^{-ip\cdot(x-y)}2\pi\delta\left(p^2+\mu^2\right)\Theta\left(p^0\right)\left[\frac{1}{2}\left(\frac{i\rho_1}{\mu} + \rho_2\right)(\not{p} - i\mu) + \frac{1}{2}\left(-\frac{i\rho_1}{\mu} + \rho_2\right)(\not{p} + i\mu)\right]
\end{aligned} \tag{38}$$

We observe the appearance of positive and negative mass fermion propagators. Hence the Feynman propagator is:

$$\begin{aligned}
\langle 0|T\psi(x)\,\bar{\psi}(y)|0\rangle &= \int_0^\infty d\mu^2\ \int \frac{d^4p}{(2\pi)^4}\ e^{-ip\cdot(x-y)}\frac{\frac{1}{2}\left(\frac{\rho_1}{\mu} - i\rho_2\right)(\not{p} - i\mu) + \frac{1}{2}\left(-\frac{\rho_1}{\mu} - i\rho_2\right)(\not{p} + i\mu)}{p^2 + \mu^2 - i\epsilon} \\
&= \int_0^\infty d\mu^2\ \int \frac{d^4p}{(2\pi)^4}\ e^{-ip\cdot(x-y)}\frac{\frac{1}{2}\left(\frac{\rho_1}{\mu} - i\rho_2\right)(\not{p} - i\mu) + \frac{1}{2}\left(-\frac{\rho_1}{\mu} - i\rho_2\right)(\not{p} + i\mu)}{p^2 + \mu^2 - i\epsilon} \\
&= \int_0^\infty d\mu^2\ \int \frac{d^4p}{(2\pi)^4}\ e^{-ip\cdot(x-y)}\left[\frac{\frac{1}{2}\left(\frac{i\rho_1}{\mu} + \rho_2\right)}{i\not{p} - \mu + i\epsilon} + \frac{\frac{1}{2}\left(-\frac{i\rho_1}{\mu} + \rho_2\right)}{i\not{p} + \mu - i\epsilon}\right] \\
&= \int \frac{d^4p}{(2\pi)^4}\ e^{-ip\cdot(x-y)}\frac{Z}{i\not{p} - m + i\epsilon} + \int_{m^2_{\text{threshold}}}^\infty d\mu^2\ \int \frac{d^4p}{(2\pi)^4}\ e^{-ip\cdot(x-y)}\left[\frac{\frac{1}{2}\left(\frac{i\sigma_1}{\mu} + \sigma_2\right)}{i\not{p} - \mu + i\epsilon} + \frac{\frac{1}{2}\left(-\frac{i\sigma_1}{\mu} + \sigma_2\right)}{i\not{p} + \mu - i\epsilon}\right]
\end{aligned} \tag{39}$$

where σs denote the spectral density of multi-particle states. The interpretation of this result is straightforward, we get contributions to the spectral density not only from particles, but also from ant-particles represented by the negative mass propagator in our formula.

Using the canonical anticommutator of the spinor field one can prove that

$$1 = Z + \int_{4m^2}^\infty d\mu^2\ \sigma_2\left(\mu^2\right)\ . \tag{40}$$

4. (a) The self energy is read off from the diagram and is given by

$$i\Pi\left(p^2\right) = \frac{1}{2}(ig)^2\int \frac{d^6k_1}{(2\pi)^6}\frac{d^6k_2}{(2\pi)^6}(2\pi)^6\,\delta\left(k_1 + k_2 - p\right)\frac{-i}{k_1^2 + m^2 - i\epsilon}\frac{-i}{k_2^2 + m^2 - i\epsilon} \tag{41}$$

The Cutkosky rules tell us how to put the internal particles on shell–we replace propagators with delta functions and appropriate factors of i and 2π to get the imaginary part

$$\mathrm{Im}\,\Pi\left(p^2\right) = \frac{1}{4}g^2\int\frac{d^6k_1}{\left(2\pi\right)^6}\frac{d^6k_2}{\left(2\pi\right)^6}\left(2\pi\right)^6\delta\left(k_1+k_2-p\right)\left(2\pi\right)\delta\left(k_1^2+m^2\right)\theta\left(k_1^0\right)\left(2\pi\right)\delta\left(k_2^2+m^2\right)\theta\left(k_2^0\right) \quad (42)$$

The delta functions and the theta functions we recognize as being part of the Lorentz invariant measure. We know that we can equivalently write things as

$$\mathrm{Im}\,\Pi\left(p^2\right) = \frac{1}{4}g^2\int\frac{d^5k_1}{\left(2\pi\right)^5 2E_{k_1}}\frac{d^5k_2}{\left(2\pi\right)^5 2E_{k_2}}\left(2\pi\right)^6\delta\left(k_1+k_2-p\right) \quad (43)$$

with the χ particles on shell. We can immediately do one of these integrals using the delta function–we also boost to a frame where $\vec{p}=0$ (note, though, that we have not yet put this particle on shell). Thus we get

$$\mathrm{Im}\,\Pi\left(p^2\right) = \frac{g^2}{16\left(2\pi\right)^4}\int\frac{d^5k_1}{m^2+\vec{k}_1^2}\delta\left(E_\phi - 2\sqrt{m^2+\vec{k}_1^2}\right) \quad (44)$$

Now use $d^5k = \frac{8\pi^2}{3}k_1^4dk_1$ and define $E\equiv 2\sqrt{m^2+\vec{k}_1^2}$. We get

$$\mathrm{Im}\,\Pi\left(p^2\right) = \frac{g^2\pi^2}{6\left(2\pi\right)^4}\int\frac{k_1^3dE}{E}\delta\left(E_\phi - E\right) \quad (45)$$

$$= \frac{g^2\pi^2}{6\left(2\pi\right)^4}\left(\frac{1}{4}-\frac{m^2}{E_\phi^2}\right)^{3/2}E_\phi^2\;\Theta\left(E_\phi-2m\right) \quad (46)$$

Since we know the result is Lorentz invariant, we can write this for a general frame

$$\mathrm{Im}\,\Pi\left(k^2\right) = -\frac{\alpha\pi}{12}k^2\left(1+\frac{4m^2}{k^2}\right)^{3/2}\;\Theta\left(\sqrt{-k^2}-2m\right) \quad (47)$$

with $\alpha\equiv g^2/\left(4\pi\right)^3$. Let's compare this to the answer we got in class by just directly taking the imaginary part of that answer. In class, we got

$$\Pi\left(k^2\right) = \frac{\alpha}{2}\int_0^1 dx\,D\log\left(\frac{D}{|D_0|}\right) - \frac{1}{2}\alpha\left(k^2+M^2\right) \quad (48)$$

with

$$D = x\left(1-x\right)k^2+m^2-i\epsilon \quad (49)$$

$$D_0 = -x\left(1-x\right)M^2+m^2 \quad (50)$$

This expression is imaginary(there is a log of a negative number) when $D<0$. Solving for the quadratic equation for this inequality, we get that the expression is imaginary when x is between x_- and x_+ with

$$x_\pm = \frac{1}{2}\pm\frac{1}{2}\sqrt{1+\frac{4m^2}{k^2}} \quad (51)$$

Note that this works only for $k^2<-4m^2$. (The criterion of real roots is $k^2<-4m^2$ or $k^2>0$. The latter gives $x_\pm$ outside $[0,1]$ and D is real on this interval.) In this range, the imaginary part of the log is $-i\pi$ (the $i\epsilon$ prescription tells us to come from the lower half plane). Thus

$$\mathrm{Im}\,\Pi\left(k^2\right) = -\frac{\pi\alpha}{2}\int_{x_-}^{x_+}dx\,D\;\Theta\left(\sqrt{-k^2}-2m\right) \quad (52)$$

$$= -\frac{\pi\alpha}{2}\int_{x_-}^{x_+}dx\;\left(x\left(1-x\right)k^2+m^2\right)\;\Theta\left(\sqrt{-k^2}-2m\right) \quad (53)$$

$$= -\frac{\alpha\pi}{12}k^2\left(1+\frac{4m^2}{k^2}\right)^{3/2}\;\Theta\left(\sqrt{-k^2}-2m\right) \quad (54)$$

which agrees indeed.

(b) In lecture we obtained that

$$\Gamma = \frac{\alpha\pi}{12} M \left(1 - \frac{4m^2}{M^2}\right)^{3/2} \tag{55}$$

is the decay rate of $\phi \to 2\chi$ if $M > 2m$. We see that equation (43) (when ϕ is put on shell) is exactly the tree level decay rate multiplied by M. Recall that at tree level $|T|^2 = g^2$ and hence the phase space volume gives us the correct result. Also by setting $k^2 = -M^2$ in (47) we get

$$\mathrm{Im}\,\Pi\left(-M^2\right) = \frac{\alpha\pi}{12} M^2 \left(1 - \frac{4m^2}{M^2}\right)^{3/2} = M\Gamma \tag{56}$$

(c) We have

$$iG_F(p) = \frac{1}{p^2 + M^2 - \mathrm{Re}\,\Pi(p^2) - i\mathrm{Im}\,\Pi(p^2) - i\epsilon} \tag{57}$$

We will Taylor expand the denominator around $p^2 = -M^2$. For example

$$\Pi\left(p^2\right) = \Pi\left(-M^2\right) + \left(p^2 + M^2\right) \frac{d\Pi\left(-M^2\right)}{dp^2} + \ldots \tag{58}$$

$$= i\mathrm{Im}\,\Pi\left(-M^2\right) + i\left(p^2 + M^2\right) \frac{d\mathrm{Im}\,\Pi\left(-M^2\right)}{dp^2} + \ldots \tag{59}$$

by the renormalization conditions. Then we have

$$iG_f(p) \approx \frac{1}{p^2 + M^2 - i\mathrm{Im}\,\Pi(-M^2) - i(p^2 + M^2)\frac{d\mathrm{Im}\,\Pi(-M^2)}{dp^2}} \tag{60}$$

$$\approx \frac{1}{-iM\Gamma + (p^2 + M^2)\left(1 - i\frac{d\mathrm{Im}\,\Pi(-M^2)}{dp^2}\right)} \tag{61}$$

$$\approx \frac{1 + i\frac{d\mathrm{Im}\,\Pi\left(-M^2\right)}{dp^2}}{p^2 + M^2 - iM\Gamma} \approx \frac{Z}{p^2 + M^2 - iM\Gamma} \tag{62}$$

$$Z = 1 + i\frac{d\mathrm{Im}\,\Pi\left(-M^2\right)}{dp^2} \tag{63}$$

Here we have used the fact that, to $O\left(g^2\right)$ order $\Gamma = Z\Gamma$ and so this step introduces higher order error. Taking the derivative of our result in (a), we find

$$i\frac{d\mathrm{Im}\,\Pi\left(p^2\right)}{dp^2}|_{p^2-M^2} = -\frac{i\alpha\pi}{12}\left(1 - \frac{4m^2}{M^2}\right)^{1/2}\left(1 + \frac{2m^2}{M^2}\right) \tag{64}$$

Thus

$$Z = 1 + i\frac{d\mathrm{Im}\,\Pi\left(-M^2\right)}{dp^2} = 1 - \frac{i\alpha\pi}{12}\left(1 - \frac{4m^2}{M^2}\right)^{1/2}\left(1 + \frac{2m^2}{M^2}\right) \tag{65}$$

(d) We are interested in computing the integral

$$iG_F(t, \vec{p}) = \int \frac{d\omega}{2\pi} \frac{-Ze^{-i\omega t}}{(\omega - \omega_0)(\omega + \omega_0)} \tag{66}$$

with

$$\omega_0 = \sqrt{\vec{p}^2 + M^2 - iM\Gamma} \approx \sqrt{\vec{p}^2 + M^2}\left(1 - \frac{iM\Gamma}{2\left(\vec{p}^2 + M^2\right)^{1/2}}\right) \tag{67}$$

To evaluate the integral, we close the contour in the lower half plane and evaluate the residue at $\omega = \omega_0$. (The clockwise contour introduces an extra minus sign.) Thus

$$\int \frac{d\omega}{2\pi} \frac{-Ze^{-i\omega t}}{(\omega - \omega_0)(\omega + \omega_0)} = \frac{iZe^{-i\omega_0 t}}{2\omega_0} \approx \left(\frac{iZe^{-iE_p t}}{2E_p}\right) e^{-\frac{M\Gamma}{2E_p}t} \sim e^{-\frac{M\Gamma}{2E_p}t} \tag{68}$$

with $E_p = \sqrt{\vec{p}^2 + M^2}$ and $\sim$ indicating the asymptotic decay of the modulus. For a particle at rest, $E_p = M$, and

$$G_F \sim e^{-\frac{\Gamma}{2}t} \tag{69}$$

G_F is intuitively interpreted as an amplitude:

$$G_F(t, \vec{p}) = \langle 0|\phi(t, \vec{p})\phi(0, -\vec{p})|0\rangle = \left(\frac{iZ}{2E_p}\right) {}_{\text{"}out\text{"}}\langle \vec{p}, t|\vec{p}, t = 0\rangle_{\text{"}in\text{"}} \ , \tag{70}$$

where the states cannot be interpreted as asymptotic states, as they decay. (We normalized the states as $\langle \vec{p}|\vec{q}\rangle = (2\pi)^5\delta^{(5)}(\vec{p}-\vec{q})$ to make the matching of the two formula more suggestive.) Nevertheless, it is clear that G_F gives the overlap between an initial ϕ particle and a ϕ particle t times from there. The physical probability will be proportional to the amplitude squared and will go like $\sim e^{-\Gamma t}$ which indeed looks like a particle decaying with rate Γ. For a more general particle in motion, since $E_p \equiv \gamma M$, with γ the Lorentz factor, we get the amplitude squared goes like $e^{-\frac{\Gamma}{\gamma}t}$ which is the correct formula for a decaying, Lorentz boosted particle.

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