

Problems 2

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1. Majorana Mass and See-Saw Mechanism

a. Let ψ be a right-handed fermion field. Using 4-component fermion notation, and some stated convention for Dirac matrices, construct the “Majorana” mass term

$$\mathcal{L}_{\text{Majorana}} = -M\overline{C\psi}\psi, \quad (1)$$

where C is the charge conjugation operator. (Of course, it is implicit that you must add the Hermitean conjugate.) Now simplify it by using the right-hand projector to express ψ in terms of its two independent components. Generalize to several “flavors” of neutrino and a mass matrix M_j^i , i.e.

$$\mathcal{L}_{\text{Majorana}} = -M_{ij}\overline{C\psi^i}\psi^j. \quad (2)$$

Does M have to be symmetric? [Hint: Use fermi statistics, or equivalently anticommutativity of the fields.]

b. Now let there be an equal number (say three) of left-handed fermion fields η^i . Assume that these do *not* have non-zero Majorana mass terms to begin with, but are allowed to have “normal” mass terms connecting them to the ψ^j , i.e.

$$\mathcal{L}_{\text{normal}} = -m_j^i\overline{\eta^i}\psi^j \quad (3)$$

Assuming $m \ll M^1$ show that the combined effect of both mass terms is, when you diagonalize the Hamiltonian at zero energy-momentum, that a Majorana mass matrix is generated for the η , of the order m^2/M . Give its precise (leading order) form.

c. You are free to make a unitary transformation involving the flavor indices of η (leaving the kinetic energy properly normalized). Prove that this is enough to diagonalize the Majorana mass matrix. Having done this, is there any residual freedom?

This problem provides the theoretical foundation for understanding neutrino masses in the context of unified gauge theories, e.g. $SO(10)$.

¹Part of the problem – what does this mean?

2. Baryon and Lepton Number Violation

- a. Show that it takes at least a dimension 6 operator to violate baryon number in the Standard Model.
- b. Show that all such dimension 6 operators conserve $B - L$.
- c. Show that there is a dimension 5 operator that violates lepton number. How does this connect to the first problem?

3. Deviant Higgs Fields

- a. Consider the possibility of scalar fields that transform as singlets under color $SU(3)$ and triplets under weak $SU(2)$. For what values of the hypercharge are these candidates for condensation²?
- b. For each of the possibilities in part a., calculate the contribution to the W and Z masses and to $Z - \gamma$ mixing. Supposing that they are present in addition to the “normal” doublet, how do observations constrain the ratio of vacuum expectation values? Be reasonably quantitative.
- c. In each case, with both an oddball and the usual doublet present, say how many physical scalar particles there are, and specify their charges.
- d. What new possibilities do such fields open up for low-dimension B and L violation?

4. Multi-Higgs, Axions

- a. Suppose that there are two Higgs doublets ϕ_1 and ϕ_2 , both with the usual assignments $(1, 2, \frac{1}{2})$. Suppose further that ϕ_1 cannot couple to right-handed charge $2/3$ quarks (our U_R) and ϕ_2 cannot couple to right-handed charge $-1/3$ quarks. (This set-up arises in supersymmetric models.) Show that if both fields condense the relation among W and Z masses and the neutral current parameter – i.e., the Weinberg angle – is just the same as for one doublet. How many physical scalars are there, and what are their electric charges?
- b. Continuing this analysis, write down the most general potential consistent with symmetry under simultaneously reversal of the signs of ϕ_1 and D_R , or ϕ_2 and U_R (this guarantees the coupling pattern assumed previously). Find the region of parameters where viable condensations occur. In this domain, calculate the spectrum of scalars and their couplings to the quarks.
- c. Suppose we have a continuous phase symmetry $\phi_1 \rightarrow e^{-i\lambda}\phi_1$, $\phi_2 \rightarrow e^{i\lambda}\phi_2$, $U_R \rightarrow e^{i\lambda}U_R$, $D_R \rightarrow e^{i\lambda}D_R$. (This set-up arises in axion models.) Show that the fermion-Higgs coupling patterns are as advertized. What are the extra restrictions on the potential, beyond those found in b.? Show that there is now a massless neutral scalar³, and calculate how it couples to quarks.

²Assuming, of course, that we want to break $SU(2) \times U(1) \rightarrow U(1)$, preserving electromagnetism

³In a quantum treatment, the symmetry will be violated – an “anomaly” – and a small mass will be generated.