8.325 Homework 3

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Due: In lecture March 22.

Problem 1) Composite Operator Renormalization

In lecture we discussed the renormalization of parameters in a Lagrangian. Often one is interested in studying a local current built from a product of fields at the same space-time point. For example, consider $J_{\mu\nu} = \bar{\psi}(0)\sigma_{\mu\nu}\psi(0)$ in QED. External operators like $J^{\mu\nu}$ may require renormalization beyond that associated with the field $\psi(0)$ (you discussed an example of this type in section). In this problem I want you to compute the anomalous dimension for $J_{\mu\nu}$. To turn this into a problem you know how to solve, consider adding a term

$$\mathcal{L}_{\rm int} = C_0^{\mu\nu}\,\bar{\psi}_0\sigma_{\mu\nu}\psi_0 \tag{1}$$

to the QED Lagrangian. By switching from bare to renormalized coefficients and fields as usual, compute the anomalous dimension for $C^{\mu\nu}$ in $\overline{\rm MS}$ to order e^2. [Hint: Remember you only need the $1/\epsilon$ poles. Compare eqs.18.10 and 18.11 in Peskin. You can use Z_ψ from Peskin without computation.] Now write $\mathcal{L}_{\rm int} = C^{\mu\nu}(\mu)J_{\mu\nu}(\mu)$ to define the renormalized current, and use the anomalous dimension for $C^{\mu\nu}(\mu)$ to find the anomalous dimension of $J_{\mu\nu}(\mu)$.

Problem 2) The decay $\pi^- \to \mu^-\bar{\nu}_\mu$

At low energy, $p^2 \ll m_W^2 = (80\,{\rm GeV})^2$ we can expand the W-boson propagator, $1/(p^2 - m_W^2) = -1/m_W^2 + \ldots$. Keeping the first term gives the effective four-fermion interaction

$$\mathcal{L}_{\rm eff} = \frac{G_F}{\sqrt{2}}\,V_{ud}\,[\bar{d}\gamma^\alpha(1-\gamma_5)u][\bar{\nu}_\mu\gamma_\alpha(1-\gamma_5)\mu] + {\rm h.c.} \tag{2}$$

which allows $\bar{u}d \to \mu^-\bar{\nu}_\mu$ and hence $\pi^- \to \mu^-\bar{\nu}_\mu$. Define the decay constant for the pion as

$$\langle 0|\bar{u}\gamma^\alpha\gamma_5 d|\pi^-(p_\pi)\rangle = -i\,f_\pi\,p_\pi^\alpha\,, \tag{3}$$

and show that the rate for $\pi^- \to \mu^-\bar{\nu}_\mu$ is

$$\Gamma = \frac{G_F^2}{8\pi}\,|V_{ud}|^2\,m_\mu^2 m_\pi f_\pi^2\Big(1 - \frac{m_\mu^2}{m_\pi^2}\Big)^2\,. \tag{4}$$

Comparing with data, and taking $V_{ud} \simeq 1$, determine a value for f_π. There are two common normalization conventions for f_π in the literature. The one in this problem agrees with the convention in the chiral Lagrangian in problem 3 below, $f_\pi = f$. The other convention is $\langle 0|J_5^{\alpha,a}|\pi^b(p_\pi)\rangle = -i\,F_\pi\,p_\pi^\alpha\,\delta^{ab}$ where $J_5^{\alpha,a} = \bar{\psi}\gamma^\alpha\gamma_5(\tau^a/2)\psi$. Derive the relation between F_π and f_π.

Problem 3) The decays $\pi^- \to \pi^0 e^- \bar{\nu}_e$ and $K^- \to \pi^0 \pi^0 e^- \bar{\nu}_e$

In lecture we derived a Feynman rule for the π^- to W^- transition by starting from the gauged SU(2) chiral Lagrangian,

$$\mathcal{L}_\chi = \frac{f^2}{8}\,\mathrm{tr}\Big[D^\mu \Sigma D_\mu \Sigma^\dagger\Big], \tag{5}$$

where $D^\mu \Sigma = \partial^\mu \Sigma + i\ell^\mu \Sigma$ and $\ell^\mu = -g_2/\sqrt{2}\, W_+^\mu T^+ + \mathrm{h.c.}$ [for simplicity we'll assume $V_{ud} = 1$ in this problem].

a) By carrying out this expansion to higher order in the pion fields derive *the amplitude* for $\pi^- \to \pi^0 e^- \bar{\nu}_e$.

b) Now generalize to the SU(3) case. Draw the tree level Feynman diagrams that can contribute to $K^- \to \pi^0 \pi^0 e^- \bar{\nu}_e$. Derive the Feynman rules that you would need to compute this amplitude (in the charged Goldstone-boson basis). Recall that $\mathcal{L}_\chi$ has a four-boson interaction. Feel free to use a program like Mathematica to take the traces. You may also use results given in lecture.