VIII.D Generic scale invariance in equilibrium systems

We live in a world full of complex spatial patterns and structures such as coastlines and river networks. There are similarly diverse temporal processes generically exhibiting "$1/f$"–noise, as in resistance fluctuations, sand flowing through an hour glass, and even in traffic and stock market movements. These phenomena lack natural length and time scales and exhibit scale invariance and self-similarity. The spacial aspects of scale invariant systems can be characterized using *fractal* geometry[1]. In this section we explore dynamical processes that can naturally result in such scale invariant patterns.

Let us assume that the system of interest is described by a scalar field $m(\mathbf{x})$, distributed with a probability $\mathcal{P}[m]$. Scale invariance can be probed by examining the correlation functions of $m(\mathbf{x})$, such as the two point correlator, $C(|\mathbf{x}-\mathbf{y}|) \equiv \langle m(\mathbf{x})m(\mathbf{y})\rangle - \langle m(\mathbf{x})\rangle \langle m(\mathbf{y})\rangle$. (It is assumed that the system has rotational and translational symmetry.) In a system with a characteristic length scale, correlations decay to zero for separations $z = |\mathbf{x}-\mathbf{y}| \gg \xi$. By contrast, if the system possesses scale invariance, correlations are homogeneous at long distances, and $\lim_{z\to\infty} C(z) \sim z^{2\chi}$.

As we have seen, in equilibrium statistical mechanics the probability is given by $\mathcal{P}_{eq} \propto \exp\left(-\beta\mathcal{H}[m]\right)$ with $\beta = (k_B T)^{-1}$. Clearly at infinite temperature there are no correlations for a finite Hamiltonian. As long as the interactions in $\mathcal{H}[m]$ are *short ranged*, it can be shown by high temperature expansions that correlations at small but finite β decay as $C(z) \propto \exp(-z/\xi)$, indicating a characteristic length-scale†. The correlation length usually increases upon reducing temperature, and may diverge if the system undergoes a continuous (critical) phase transition. At a critical transition the system is scale invariant and $C(z) \propto z^{2-d-\eta}$. However, such scale invariance is *non-generic* in the sense that it can be obtained only by precise tuning of the system to the critical temperature. Most scale invariant processes in nature do not require such precise tuning, and therefore the analogy to the critical point is not particularly instructive[2][3].

We shall frame our discussion of scale invariance by considering the dynamics of a surface, described by its height $h(\mathbf{x}, t)$. Specific examples are the distortions of a soap film or the fluctuations on the surface of water in a container. In both cases the minimum energy configuration is a flat surface (ignoring the small effects of gravity on the soap film).

† It is of course possible to generate long-range correlations with *long ranged* interactions. However, it is most interesting to find out how long-range correlations are generated from local, *short ranged* interactions.

The energy cost of small fluctuations for a soap film comes from the increased area and *surface tension* σ. Expanding the area in powers of the slope results in

$$\mathcal{H}_\sigma = \sigma \int d^d\mathbf{x} \left[\sqrt{1+(\nabla h)^2} - 1\right] \approx \frac{\sigma}{2} \int d^d\mathbf{x} \, (\nabla h)^2 \, . \tag{VIII.49}$$

For the surface of water there is an additional gravitational potential energy, obtained by adding the contributions from all columns of water as

$$\mathcal{H}_g = \int d^d\mathbf{x} \int_0^{h(\mathbf{x})} \rho g h(\mathbf{x}) = \frac{\rho g}{2} \int d^d\mathbf{x} h(\mathbf{x})^2 \, . \tag{VIII.50}$$

The total (potential) energy of small fluctuations is thus given by

$$\mathcal{H} = \int d^d\mathbf{x} \left[\frac{\sigma}{2} (\nabla h)^2 + \frac{\rho g}{2} h^2\right] , \tag{VIII.51}$$

with the second term absent for the soap film.

The corresponding Langevin equation,

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = -\mu\rho g h + \mu\sigma\nabla^2 h + \eta(\mathbf{x},t), \tag{VIII.52}$$

is linear, and can be solved by Fourier transforms. Starting with a flat interface, $h(\mathbf{x}, t = 0) = h(\mathbf{q}, t = 0) = 0$, the profile at time t is

$$h(\mathbf{x},t) = \int \frac{d^d\mathbf{q}}{(2\pi)^d} e^{-i\mathbf{q}\cdot\mathbf{x}} \int_0^t d\tau e^{-\mu(\rho g + \sigma q^2)(t-\tau)} \eta(\mathbf{q},t). \tag{VIII.53}$$

The average height of the surface, $\bar{H} = \int d^d\mathbf{x} \left\langle h(\mathbf{x},t)\right\rangle / L^d$ is zero, while its overall width is defined by

$$w^2(t,L) \equiv \frac{1}{L^d} \int d^d\mathbf{x} \left\langle h(\mathbf{x},t)^2\right\rangle = \frac{1}{L^d} \int \frac{d^d\mathbf{q}}{(2\pi)^d} \left\langle |h(\mathbf{q},t)|^2\right\rangle , \tag{VIII.54}$$

where L is the linear size of the surface. Similar to Eq.(VIII.33), we find that the width grows as

$$w^2(t,L) = \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{D}{\gamma(\mathbf{q})} \left(1 - e^{-2\gamma(\mathbf{q})t}\right). \tag{VIII.55}$$

There are a range of time scales in the problem, related to characteristic length scales as in Eq.(VIII.31). The shortest time scale, $t_{\min} \propto a^2/(\mu\sigma)$, is set by an atomic size a. The longest time scale is set by either the capillary length ($\lambda_c \equiv \sqrt{\sigma/\rho g}$) or the system size ($L$).

For simplicity we shall focus on the soap film where the effects of gravity are negligible and $t_{\max} \propto L^2/(\mu\sigma)$. We can now identify three different ranges of behavior in Eq.(VIII.55):

(a) For $t \ll t_{\min}$, none of the modes has relaxed since $\gamma(\mathbf{q})t \ll 1$ for all $\mathbf{q}$. Each mode grows diffusively, and

$$w^2(t,L) = \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{D}{\gamma(\mathbf{q})} 2\gamma(\mathbf{q})t \; = \; \frac{2Dt}{a^d}. \tag{VIII.56}$$

(b) For $t \gg t_{\max}$, all modes have relaxed to their equilibrium values since $\gamma(\mathbf{q})t \gg 1$ for all $\mathbf{q}$. The height fluctuations now saturate to a maximum value given by

$$w^2(t,L) = \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{D}{\mu\sigma q^2}. \tag{VIII.57}$$

The saturated value depends on the dimensionality of the surface, and in a general dimension d behaves as

$$w^2(t,L) \propto \frac{D}{\mu\sigma} \begin{cases} a^{2-d} & \text{for } d>2, \quad (\chi = 0) \\ \ln(L/a) & \text{for } d=2, \quad (\chi = 0^+) \\ L^{2-d} & \text{for } d<2, \quad (\chi = \frac{2-d}{2}) \end{cases}, \tag{VIII.58}$$

where we have defined a *roughness exponent* χ that governs the divergence of the width with system size via $\lim_{t\to\infty} w(t,L) \propto L^{\chi}$. (The symbol 0^+ is used to indicate a logarithmic divergence.) The exponent of $\chi = 1/2$ in $d=1$ indicates that the one dimensional interface fluctuates like a random walk.

(c) For $t_{\min} \ll t \ll t_{\max}$ only a fraction of the shorter length scale modes are saturated. The integrand in Eq.(VIII.55) (for $g=0$) is made dimensionless by setting $y = \mu\sigma q^2 t$, and

$$\begin{aligned} w^2(t,L) &\propto \frac{D}{\mu\sigma} \int dq\; q^{d-3} \left(1 - e^{-2\mu\sigma q^2 t}\right) \\ &\propto \frac{D}{\mu\sigma} \left(\frac{1}{\mu\sigma t}\right)^{\frac{d-2}{2}} \int_{t/t_{\max}}^{t/t_{\min}} dy\; y^{\frac{d-4}{2}} \left(1 - e^{-2y}\right). \end{aligned} \tag{VIII.59}$$

The final integral is convergent for $d<2$, and dominated by its upper limit for $d \geq 2$. The initial growth of the width is described by another exponent β, defined through $\lim_{t\to 0} w(t,L) \propto t^{\beta}$, and

$$w^2(t,L) \propto \begin{cases} \frac{D}{\mu\sigma} a^{2-d} & \text{for } d>2, \quad (\beta = 0) \\ \frac{D}{\mu\sigma} \ln(t/t_{\min}) & \text{for } d=2, \quad (\beta = 0^+) \\ \frac{D}{(\mu\sigma)^{d/2}} t^{(2-d)/2} & \text{for } d<2, \quad (\beta = (2-d)/4) \end{cases}. \tag{VIII.60}$$

The exponents χ and β also describe the height–height correlation functions which assumes the *dynamic scaling* form

$$\left\langle [h(\mathbf{x}, t) - h(\mathbf{x}', t')]^2 \right\rangle = |\mathbf{x} - \mathbf{x}'|^{2\chi} \, g\left(\frac{|t - t'|}{|\mathbf{x} - \mathbf{x}'|^z} \right). \tag{VIII.61}$$

Since equilibrium equal time correlations only depend on $|\mathbf{x} - \mathbf{x}'|$, $\lim_{y \to 0} g(y)$ should be a constant. On the other hand correlations at the same point can only depend on time, requiring that $\lim_{y \to \infty} g(y) \propto y^{2\chi/z}$, and leading to the exponent identity $\beta = \chi/z$.

This scale invariance is broken when the gravitational potential energy, is added to the Hamiltonian. The correlations now decay as $C(z) \propto \exp(-z/\lambda_c)$ for distances larger than the capillary length. What is the underlying difference between these two cases? The presence of gravity breaks the translational symmetry, $\mathcal{H}[h(\mathbf{x}) + c] = \mathcal{H}[h(\mathbf{x})]$. It is this continuous symmetry that forbids the presence of a term proportional to $\int d^d\mathbf{x} h(\mathbf{x})^2$ in the Hamiltonian and removes the corresponding length scale. (The coefficient of the quadratic term is usually referred to as a *mass* in field theoretical language.) The presence of a *continuous symmetry* is quite a general condition for obtaining *generic scale invariance* (GSI)[4]. As discussed in previous chapters, there are many low temperature phases of matter in which a continuous symmetry is spontaneously broken. The energy cost of small fluctuations around such a state must obey the *global* symmetry. The resulting excitations are the "massless" *Goldstone modes*. We already discussed such modes in connection with *magnons* in ferromagnets (with broken rotational symmetry), and *phonons* in solids (broken translational symmetry). All these cases exhibit scale invariant fluctuations.

In the realm of dynamics we can ask the more general question of whether temporal correlations, e.g. $C(|\mathbf{x} - \mathbf{x}'|, t - t') = \langle h(\mathbf{x}, t) h(\mathbf{x}', t') \rangle_c$, exhibit a characteristic time scale τ, or are homogeneous in $t - t'$. It is natural to expect that scale invariance in the spacial and temporal domains are closely interlinked. Establishing correlations at large distances requires long times as long as the system follows *local* dynamical rules (typically $(t - t') \propto |\mathbf{x} - \mathbf{x}'|^z$). Spacial scale invariance thus implies the lack of a time scale. The converse is not true as dynamics provides an additional possibility of removing time scales through a *conservation law*. We already encountered this situation in examining the model B dynamics of the surface Hamiltonian in the presence of gravity. Equation (VIII.47) indicates that, even though the long wavelength modes are massive, the relaxation time of a mode of wavenumber $\mathbf{q}$ diverges as $1/q^2$ in the $\mathbf{q} \to \mathbf{0}$ limit.

Symmetries and conservation laws are intimately linked in equilibrium systems. Consider a *local* Hamiltonian that is invariant under the symmetry $\mathcal{H}[h(\mathbf{x})+c] = \mathcal{H}[h(\mathbf{x})]$. Since $\mathcal{H}$ can only depend on ∇h and higher derivatives,

$$v = \mu F = -\mu \frac{\delta \mathcal{H}}{\delta h(\mathbf{x})} = \mu \nabla \cdot \frac{\partial \mathcal{H}}{\partial \nabla h} + \cdots \equiv \nabla \cdot \vec{j} \quad . \tag{VIII.62}$$

Even for model A dynamics, the deterministic part of the velocity is the divergence of a current and conserves $\int d^d\mathbf{x} h(\mathbf{x},t)$. The conservation is only statistical and locally broken by the non-conserved noise in model A. The above result is a consequence of *Noether's theorem.*

VIII.E Non-equilibrium dynamics of open systems

We have to be cautious in applying the methods and lessons of near equilibrium dynamics to the various processes in nature which exhibit generic scale invariance. Many such systems, such as a flowing river or a drifting cloud, are very far from equilibrium. Furthermore, they are open and extended systems constantly exchanging particles and constituents with their environment. It is not clear that there is any Hamiltonian that governs the dynamics of such processes and hence the traditional approach presented earlier is not necessarily appropriate. However, the robust self-similar correlations observed in these systems[1] suggests that they can be described by stationary scale invariant probability distributions. This section outlines a general approach to the dynamics of open and extended systems that is similar in spirit to the construction of effective coarse–grained field theories described in Chapter 2. Let us again consider the dynamics of a static field, $h(\mathbf{x},t)$:

1. The starting point in equilibrium statistical mechanics is the Hamiltonian $\mathcal{H}[h]$. Landau's prescription is to include in $\mathcal{H}$ *all terms consistent with the symmetries* of the problem. The underlying philosophy is that in a generic situation an allowed term is present, and can only vanish by accident. In the case of non-equilibrium dynamics we shall assume that the *equation of motion* is the fundamental object of interest. Over sufficiently long time scales, inertial terms ($\propto \partial_t^2 h$) are irrelevant in the presence of dissipative dynamics, and the evolution of h is governed by

$$\partial_t h(\mathbf{x},t) = \overbrace{v\left[h(\mathbf{x},t)\right]}^{\text{deterministic}} + \overbrace{\eta(\mathbf{x},t)}^{\text{stochastic}} \quad . \tag{VIII.63}$$

2. If the interactions are short ranged, the velocity at $(\mathbf{x}, t)$ depends only on $h(\mathbf{x}, t)$ and a few derivatives evaluated at $(\mathbf{x}, t)$, i.e.

$$v(\mathbf{x}, t) = v\big(h(\mathbf{x}, t), \nabla h(\mathbf{x}, t), \cdots\big). \tag{VIII.64}$$

3. We must next specify the functional form of deterministic velocity, and the correlations in noise. Generalizing Landau's prescription, we assume that all terms consistent with the underlying *symmetries and conservation laws* will generically appear in v. The noise, $\eta(\mathbf{x}, t)$, may be conservative or non-conservative depending on whether there are only internal rearrangements, or external inputs and outputs.

Corollary: Note that with these set of rules there is no reason for the velocity to be derivable from a potential ($v \neq -\hat{\mu}\delta\mathcal{H}/\delta h$), and there is no fluctuation–dissipation condition ($\hat{D} \neq \hat{\mu}k_BT$). It is even possible for the deterministic velocity to be conservative, while the noise is not. Thus various familiar results of near equilibrium dynamics may no longer hold.

As an example consider the flow of water along a river (or traffic along a highway). The deterministic part of the dynamics is conservative (the amount of water, or the number of cars is unchanged). Hence the velocity is the divergence of a current, $v = -\nabla\vec{j}\,[h]$. The current, $\vec{j}$, is a vector, and must be constructed out of the other two vectorial quantities in the problem: the gradient operator ∇, and the average transport direction $\hat{t}$. (The unit vector $\hat{t}$ points along the direction of current or traffic flow.) The lowest order terms in the expansion of the current give

$$-\vec{j} = \hat{t}(\alpha h - \frac{\lambda}{2}h^2 + \cdots) + \nu_1\nabla h + \nu_2\hat{t}(\hat{t}\cdot\nabla)h + \cdots. \tag{VIII.65}$$

The components of current parallel and perpendicular to the net flow are

$$\begin{cases} -j_\parallel = \alpha h - \frac{\lambda}{2}h^2 + (\nu_1 + \nu_2)\partial_\parallel h + \cdots \\ -\vec{j}_\perp = \nu_1\vec{\partial}_\perp h + \cdots \end{cases}. \tag{VIII.66}$$

The resulting equation of motion is

$$\frac{\partial h(\mathbf{x}, t)}{\partial t} = \partial_\parallel(\alpha h - \frac{\lambda}{2}h^2) + (\nu_1 + \nu_2)\partial_\parallel^2 h + \nu_1\partial_\perp^2 h + \cdots + \eta(\mathbf{x}, t). \tag{VIII.67}$$

In the absence of external inputs and outputs (no rain, drainage, or exits), the noise is also conservative, with correlations,

$$\langle\eta(\mathbf{q}, t)\rangle = 0, \quad \text{and} \quad \langle\eta(\mathbf{q}, t)\eta(\mathbf{q}', t')\rangle = 2(D_\parallel q_\parallel^2 + D_\perp q_\perp^2)\,\delta(t - t')(2\pi)^d\delta^d(\mathbf{q} + \mathbf{q}'). \tag{VIII.68}$$

Note that the symmetries of the problem allow different noise correlations parallel and perpendicular to the net flow.

Equations (VIII.67) and (VIII.68) define a *driven diffusion system* (DDS)[5],[6]. The first term in Eq.(VIII.67) can be eliminated by looking at fluctuations in a moving frame,

$$h(x_\parallel, \mathbf{x}_\perp, t) \to h(x_\parallel - \alpha t, \mathbf{x}_\perp, t), \tag{VIII.69}$$

and shall be ignored henceforth. Neglecting the non-linear terms, these fluctuations satisfy the anisotropic noisy diffusion equation

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = \nu_\parallel \partial_\parallel^2 h + \nu_\perp \partial_\perp^2 h + \eta(\mathbf{x},t), \tag{VIII.70}$$

where $\nu_\parallel = \nu_1 + \nu_2$ and $\nu_\perp = \nu_1$. The Fourier modes now relax with characteristic times,

$$\tau(\mathbf{q}) = \frac{1}{\nu_\parallel q_\parallel^2 + \nu_\perp q_\perp^2}. \tag{VIII.71}$$

Following the steps leading to Eq.(VIII.48),

$$\lim_{t\to\infty} \left\langle |h(\mathbf{q},t)|^2 \right\rangle = \frac{D_\parallel q_\parallel^2 + D_\perp q_\perp^2}{\nu_\parallel q_\parallel^2 + \nu_\perp q_\perp^2}. \tag{VIII.72}$$

The stationary correlation functions† in real space now behave as[4][6]

$$\begin{aligned}\left\langle (h(\mathbf{x}) - h(\mathbf{0}))^2 \right\rangle &= \int \frac{d^{d-1}\mathbf{q}_\perp dq_\parallel}{(2\pi)^d} \left(2 - 2\cos(q_\parallel x_\parallel + \mathbf{q}_\perp \cdot \mathbf{x}_\perp)\right) \frac{D_\parallel q_\parallel^2 + D_\perp q_\perp^2}{\nu_\parallel q_\parallel^2 + \nu_\perp q_\perp^2} \\ &\propto \left(\frac{D_\perp}{\nu_\perp} - \frac{D_\parallel}{\nu_\parallel}\right) \sqrt{\frac{\nu_\parallel^{d-1}\nu_\perp}{\left(\nu_\perp x_\parallel^2 + \nu_\parallel x_\perp^2\right)^d}}\end{aligned}. \tag{VIII.73}$$

Note that these correlations are spatially extended and scale invariant. This is again a consequence of the conservation law. Only in the special case where $D_\perp/\nu_\perp = D_\parallel/\nu_\parallel$ is the Einstein relation ($D(\mathbf{q}) \propto \nu(\mathbf{q})$) satisfied, and the fluctuations become uncorrelated ($C(\mathbf{x}) \propto \delta^d(\mathbf{x}) D/\nu$). The results then correspond to model B dynamics with a Hamiltonian $\mathcal{H} \propto \int d^d\mathbf{x} h^2$. This example illustrates the special nature of near equilibrium dynamics. The fluctuation–dissipation condition is needed to ensure approach to the equilibrium state

† In non-equilibrium circumstances we shall use the term *stationary* to refer to behavior at long times.

where there is typically no scale invariance. On the other hand, removing this restriction leads to GSI in a conservative system. ($D_\perp/\nu_\perp \neq D_\parallel/\nu_\parallel$ is like having two different temperatures parallel and perpendicular to the flow.)

Let us now break the conservation law stochastically by adding random inputs and outputs to the problem (in the form of rain, drainage, or exits). The properties of the noise are now modified to

$$\langle \eta(\mathbf{x},t)\rangle = 0, \quad \text{and} \quad \langle \eta(\mathbf{x},t)\eta(\mathbf{x}',t')\rangle = 2D\delta(x_\parallel - x'_\parallel)\delta^{d-1}(\mathbf{x}_\perp - \mathbf{x}'_\perp)\delta(t-t'). \quad \text{(VIII.74)}$$

In the stationary state,

$$\lim_{t\to\infty}\left\langle |h(\mathbf{q},t)|^2\right\rangle = \frac{D}{\nu_\parallel q_\parallel^2 + \nu_\perp q_\perp^2}, \quad \text{(VIII.75)}$$

in Fourier space, and

$$\left\langle (h(\mathbf{x}) - h(\mathbf{0}))^2\right\rangle \propto D\left(\nu_\perp x_\parallel^2 + \nu_\parallel x_\perp^2\right)^{\frac{2-d}{2}}, \quad \text{(VIII.76)}$$

in real space. Except for the anisotropy, this is the same result as in Eq.(VIII.61), with $\chi = (2-d)/2$.

How are the results modified by the nonlinear term $(-\lambda\partial_\parallel h^2/2)$ in Eq.(VIII.67)? We first perform a simple *dimensional analysis* by rescaling lengths and time. Allowing for anisotropic scalings, we set $x_\parallel \to b x_\parallel$, accompanied by $t \to b^z t$, $\vec{x}_\perp \to b^\zeta \vec{x}_\perp$, and $h \to b^\chi h$. Eq.(VIII.67) is now modified to

$$b^{\chi - z}\frac{\partial h}{\partial t} = \nu_\parallel b^{\chi-2}\partial_\parallel^2 h + \nu_\perp b^{\chi - 2\zeta}\partial_\perp^2 h - \frac{\lambda}{2}b^{2\chi-1}\partial_\parallel h^2 + b^{-z/2-(d-1)\zeta/2-1/2}\eta, \quad \text{(VIII.77)}$$

where Eq.(VIII.74) has been used to determine the scaling of η. We thus identify the bare scalings for these parameters as

$$\begin{cases} \nu_\parallel \to b^{z-2}\nu_\parallel \\ \nu_\perp \to b^{z-2\zeta}\nu_\perp \\ \lambda \to b^{\chi+z-1}\lambda \\ D \to b^{z-2\chi-\zeta(d-1)-1}D \end{cases}. \quad \text{(VIII.78)}$$

In the absence of λ, the parameters can be made *scale invariant* (i.e. independent of b), by the choice of $\zeta_0 = 1$, $z_0 = 2$, and $\chi_0 = (2-d)/2$, as encountered before. However, with this choice, a small λ will grow under rescaling as

$$\lambda \to b^{y_0}\lambda, \quad \text{with} \quad y_0 = \frac{4-d}{2}. \quad \text{(VIII.79)}$$

Since the non-linearity grows larger under scaling it cannot be ignored in dimensions $d < 4$.

Equations (VIII.79) constitute a simple renormalization group (RG) that is valid close to the *fixed point* (scale invariant equation) corresponding to a linearized limit. To calculate the RG equations at finite values of nonlinearity in general requires a perturbative calculation. Sometimes there are exact *non–renormalization conditions* that simplify the calculation and lead to exponent identities. Fortunately there are enough such identities for Eq.(VIII.78) that the three exponents can be obtained *exactly*.

1. As the nonlinearity is proportional to $q_{\|}$ is Fourier space, it does not generate under RG any contributions that can modify $\nu_{\perp}$. The corresponding 'bare' scaling of $\nu_{\perp}$ in Eqs.(VIII.78) is thus always valid; its fixed point leads to the exponent identity $z - 2\zeta = 0$.
2. As the nonlinearity is in the conservative part, it does not renormalize the strength of the non–conservative noise. The non–renormalization of D leads to the exponent identity $z - 2\chi - \zeta(d-1) - 1 = 0$. (This condition has a natural counterpart in equilibrium model B dynamics, $z - 2\chi - d - 2 = 0$, leading to the well known relation, $z = 4 - \eta$.)
3. Eq.(VIII.67) is invariant under an infinitesimal reparameterization $x_{\|} \to x_{\|} - \delta\lambda t$, $t \to t$, if $h \to h + \delta$. Note that the parameter λ appears both as the coefficient of the non-linearity in Eq.(VIII.67) and as an invariant factor relating the $x_{\|}$ and h reparameterizations. Hence any renormalization of the driven diffusion equation that preserves this symmetry must leave the coefficient λ unchanged, leading to the exponent identity $z + \chi - 1 = 0$.

The remaining parameter, $\nu_{\|}$ does indeed follow a non-trivial evolution under RG. However, the above three exponent identities are sufficient to give the exact exponents in all dimensions $d \leq 4$ as[2]

$$\chi = \frac{1-d}{7-d}, \quad z = \frac{6}{7-d}, \quad \zeta = \frac{3}{7-d}. \tag{VIII.80}$$

VIII.F Dynamics of a growing surface

The rapid growth of crystals by deposition, or molecular beam epitaxy, is an important technological process. It also provides the simplest example of a non-equilibrium evolution process[7]. We would like to understand the dynamic scaling of fluctuations inherent to this type of growth. To construct the dependence of the local, deterministic velocity, v, on the surface height, $h(\mathbf{x}, t)$, note that:

(1) As long as the rearrangements of particles on the surface can result in holes and vacancies, there is no conservation law.

(2) There is a *translation symmetry*, $v[h(\mathbf{x})+c]=v[h(\mathbf{x})]$, implying that v depends only on gradients of $h(\mathbf{x})$.

(3) For simplicity, we shall focus on *isotropic* surfaces, in which all directions in $\mathbf{x}$ are equivalent[8].

(4) There is no up–down symmetry, i.e. $v[h(\mathbf{x})]\neq -v[-h(\mathbf{x})]$. The absence of such symmetry allows addition of terms of both parities.

With these conditions, the lowest order terms in the equation of motion give[9],

$$\frac{\partial h(\mathbf{x},t)}{\partial t}=u+\nu\nabla^2 h+\frac{\lambda}{2}\left(\nabla h\right)^2+\cdots+\eta(\mathbf{x},t), \tag{VIII.81}$$

with the non-conservative noise satisfying the correlations in Eq.(VIII.74).

In Eq.(VIII.81), u is related to the average growth velocity. In fact, the coefficients of all even terms must be proportional to u as they all vanish in the symmetric case with no preferred growth direction. The constant u is easily removed by transforming to a moving frame, $h\to h-ut$, and will be ignored henceforth. The first non-trivial term is the nonlinear contribution, $\lambda(\nabla h)^2/2$. Geometrically this term can be justified by noting that growth by addition of particles proceeds through a parallel transport of the surface gradient in the normal direction. (See the inset to Fig. 9.1.) This term cannot be generated from the variations of any Hamiltonian, i.e. $v\neq-\mu\delta\mathcal{H}[h]/\delta h$. Thus, contrary to the equilibrium situation (Noether's theorem), the translational symmetry does not imply a conservation law, $v\neq-\nabla j$.

Further evidence of the relevance of Eq.(VIII.81) to growth phenomena is provided by examining *deterministic growth*. Consider a slow and uniform snowfall, on an initial profile which at $t=0$ is described by $h_0(\mathbf{x})$. The nonlinear equation can in fact be *linearized* with the aid of a "Cole–Hopf" transformation,

$$W(\mathbf{x},t)=\exp\left[\frac{\lambda}{2\nu}h(\mathbf{x},t)\right]. \tag{VIII.82}$$

The function $W(\mathbf{x},t)$ evolves according to the diffusion equation with *multiplicative noise*,

$$\frac{\partial W(\mathbf{x},t)}{\partial t}=\nu\nabla^2 W+\frac{\lambda}{2\nu}\,W\eta(\mathbf{x},t). \tag{VIII.83}$$

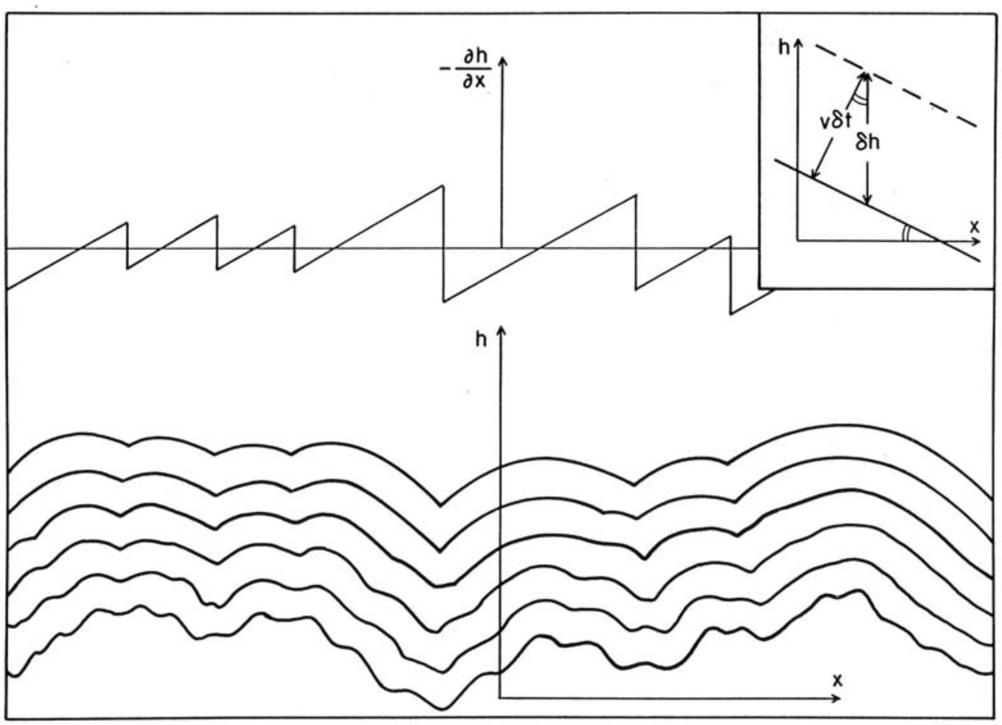
In the absence of noise, $\eta(\mathbf{x},t) = 0$, Eq.(VIII.83) can be solved subject to the initial condition $W(\mathbf{x}, t = 0) = \exp[\lambda h_0(\mathbf{x})/2\nu]$, and leads to the growth profile,

$$h(\mathbf{x},t) = \frac{2\nu}{\lambda} \ln \left\{ \int d^d\mathbf{x}' \exp\left[-\frac{|\mathbf{x}-\mathbf{x}'|^2}{2\nu t} + \frac{\lambda}{2\nu} h(\mathbf{x},t) \right] \right\}. \tag{VIII.84}$$

It is instructive to examine the $\nu \to 0$ limit, which is indeed appropriate to snow falls since there is not much rearrangement after deposition. In this limit, the integral in Eq.(VIII.84) can be performed by the saddle point method. For each $\mathbf{x}$ we have to identify a point $\mathbf{x}'$ which maximizes the exponent, leading to a collection of paraboloids described by

$$h(\mathbf{x},t) = \max \left\{ h_0(\mathbf{x}') - \frac{|\mathbf{x}-\mathbf{x}'|^2}{2\lambda t} \right\}_{\mathbf{x}'}. \tag{VIII.85}$$

Such parabolic sequences are quite common in many layer by layer growth processes in nature, from biological to geological formations. The patterns for $\lambda = 1$ are identical to those obtained by a geometrical method of Huygens, familiar from optics. The growth profile (wave front) is constructed from the outer envelop of circles of radius λt drawn from all points on the initial profile. The nonlinearity in Eq.(VIII.81) algebraically captures this process of expanding wave fronts.



VIII.1. Deterministic growth according to eq,(VIII.81) leads to a pattern of coarsening paraboloids. In one dimension, the slope of the interface forms ‘shock fronts.’ Inset depicts projection of lateral growth on the vertical direction.

As growth proceeds, the surface smoothens by the *coarsening* of the parabolas. What is the typical size of these features at time t? In maximizing the exponent in Eq.(VIII.85), we have to balance a reduction $|\mathbf{x}-\mathbf{x}'|^2/2\lambda t$, by a possible gain from $h_0(\mathbf{x}')$ in selecting a point away from $\mathbf{x}$. The final scaling is controlled by the roughness of the initial profile. Let us assume that the original pattern is a *self-affine fractal* of roughness χ, i.e.

$$\overline{|h_0(\mathbf{x})-h_0(\mathbf{x}')|} \sim |\mathbf{x}-\mathbf{x}'|^{\chi}. \tag{VIII.86}$$

(According to Mandelbrot, $\chi \approx 0.7$ for mountains[1].) Balancing the two terms in Eq.(VIII.85) gives

$$(\delta x)^{\chi} \sim \frac{(\delta x)^2}{t} \quad \Longrightarrow \quad \delta x \sim t^{1/z}, \quad \text{with} \quad z+\chi=2. \tag{VIII.87}$$

For example, if the initial profile is like a random walk in $d=1$, $\chi=1/2$, and $z=3/2$. This leads to the spreading of information along the profile by a process that is faster than diffusion, $\delta x \sim t^{2/3}$.

Note that the slope, $\vec{v}(\mathbf{x},t)=-\lambda\vec{\nabla}h(x,t)$, satisfies the equation,

$$\frac{D\vec{v}(\mathbf{x},t)}{Dt} \equiv \frac{\partial \vec{v}}{\partial t}+\vec{v}\cdot\vec{\nabla}\vec{v}=\nu\nabla^2\vec{v}-\lambda\nabla\eta. \tag{VIII.88}$$

The above is the Navier–Stokes equation for the velocity of a fluid of viscosity ν, which is being randomly stirred by a conservative force[10], $\vec{f}=-\lambda\nabla\eta$. However, the fluid is vorticity free since

$$\vec{\Omega}=\vec{\nabla}\times\vec{v}=-\lambda\nabla\times\nabla h=0. \tag{VIII.89}$$

This is the *Burgers' equation*[11], which provides a simple example of the formation of shock waves in a fluid. The gradient of Eq.(VIII.85) in $d=1$ gives a saw tooth pattern of shocks which coarsen in time. Further note that in $d=1$ Eq.(VIII.88) is also equivalent to the driven diffusion equation of (VIII.67), with $\vec{v}$ playing the role of h.

To study stochastic roughening in the presence of the nonlinear term, we carry out a scaling analysis as in Eq.(VIII.77). Under the scaling $\mathbf{x}\to b\mathbf{x}$, $t\to b^z t$, and $h\to b^{\chi}h$, Eq.(VIII.81) transforms to

$$b^{\chi-z}\frac{\partial h}{\partial t}=\nu b^{\chi-2}\nabla^2 h+\frac{\lambda}{2}b^{2\chi-2}\left(\nabla h\right)^2+\eta\left(b\mathbf{x},b^z t\right). \tag{VIII.90}$$

The correlations of the transformed noise, $\eta'(\mathbf{x},t) = b^{z-\chi}\eta(b\mathbf{x}, b^z t)$, satisfy

$$\begin{aligned}\langle \eta'(\mathbf{x},t)\eta'(\mathbf{x}',t')\rangle =& b^{2z-2\chi}\, 2D\, \delta^d(\mathbf{x}-\mathbf{x}')b^{-d}\delta(t-t')b^{-z} \\ =& b^{z-d-2\chi}\, 2D\, \delta^d(\mathbf{x}-\mathbf{x}')\delta(t-t')\end{aligned} . \qquad \text{(VIII.91)}$$

Following this scaling the parameters of Eq.(VIII.81) are transformed to

$$\begin{cases} \nu \longrightarrow b^{z-2}\nu \\ \lambda \longrightarrow b^{\chi+z-2}\lambda \\ D \longrightarrow b^{z-2\chi-d}D \end{cases} . \qquad \text{(VIII.92)}$$

For $\lambda = 0$, the equation is made scale invariant upon the choice of $z_0 = 2$, and $\chi_0 = (2-d)/2$. Close to this linear fixed point, λ scales to $b^{z_0+\chi_0-2}\lambda = b^{(2-d)/2}\lambda$, and is a relevant operator for $d < 2$. In fact a perturbative dynamic renormalization group suggests that it is *marginally relevant* at $d = 2$, and that in all dimensions a sufficiently large λ leads to new scaling behavior. (This will be discussed further in the next chapter.)

Are there any non–renormalization conditions that can help in identifying the exponents of the full nonlinear stochastic equation? Note that since Eqs.(VIII.81) and (VIII.88) are related by a simple transformation, they must have the same scaling properties. Since the Navier–Stokes equation is derivable from Newton's laws of motion for fluid particles, it has the Galilean invariance of changing to a uniformly moving coordinate frame. This symmetry is preserved under renormalization to larger scales and requires that the ratio of the two terms on the left hand side of Eq.(VIII.88) ($\partial_t\vec{v}$ and $\vec{v}\cdot\nabla\vec{v}$) stays at unity. In terms of Eq.(VIII.81) this implies the non–renormalization of the parameter λ, and leads to the exponent identity

$$\chi + z = 2. \qquad \text{(VIII.93)}$$

Unfortunately there is no other non–renormalization condition except in $d = 1$. Following Eq.(VIII.36), we can write down a Fokker–Planck equation for the evolution of the configurational probability as,

$$\frac{\partial\mathcal{P}([h(\mathbf{x})],t)}{\partial t} = -\int d^d\mathbf{x} \frac{\delta}{\delta h(\mathbf{x})}\left[\left(\nu\nabla^2 h + \frac{\lambda}{2}(\nabla h)^2\right)\mathcal{P} - D\frac{\delta\mathcal{P}}{\delta h(\mathbf{x})}\right]. \qquad \text{(VIII.94)}$$

Since Eq.(VIII.81) was not constructed from a Hamiltonian, in general we do not know the stationary solution at long times. In $d = 1$, we make a guess and try a solution of the form

$$\mathcal{P}_0[h(x)] \propto \exp\left[-\frac{\nu}{2D}\int dx(\partial_x h)^2\right]. \qquad \text{(VIII.95)}$$

Since

$$\frac{\delta \mathcal{P}_0}{\delta h(x)} = -\partial_x \frac{\delta \mathcal{P}_0}{\delta(\partial_x h)} = \frac{\nu}{D}\,(\partial_x^2 h)\,\mathcal{P}_0, \tag{VIII.96}$$

Eq.(VIII.94) leads to,

$$\begin{aligned}\frac{\partial \mathcal{P}_0}{\partial t} &= -\int dx \frac{\delta \mathcal{P}_0}{\delta h(x)} \left(\nu \partial_x^2 h + \frac{\lambda}{2}(\partial_x h)^2 - D\frac{\nu}{D}\partial_x^2 h\right) \\ &= -\frac{\lambda}{2}\mathcal{P}_0 \int dx \frac{\nu}{D}(\partial_x^2 h)(\partial_x h)^2 = -\frac{\lambda\nu}{2D}\mathcal{P}_0 \int dx \partial_x \left(\frac{(\partial_x h)^3}{3}\right) = 0.\end{aligned} \tag{VIII.97}$$

We have thus identified the stationary state of the one dimensional equation. (This procedure does not work in higher dimensions as it is impossible to write the final result as a total derivative.) Surprisingly, the stationary distribution is the same as the one in equilibrium at a temperature proportional to D/ν. We can thus immediately identify the roughness exponent $\chi = 1/2$, which together with the exponent identity in Eq.(VIII.93) leads to $z = 3/2$, i.e. super-diffusive behavior.

The values of the exponents in the strongly non-linear regime are not known exactly in higher dimensions. However, extensive numerical simulations of growth have provided fairly reliable estimates[7]. In the physically relevant case ($d = 2$) of a surface grown in three dimensions, $\chi \approx 0.39$ and $z \approx 1.61$[12]. A rather good (but not exact) fit to the exponents in a general dimension d is the following estimate by Kim and Kosterlitz[13],

$$\chi \approx \frac{2}{d+3} \quad \text{and} \quad z \approx \frac{2(d+2)}{d+3}. \tag{VIII.98}$$

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Spring 2014