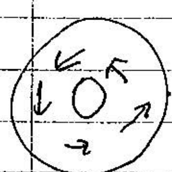
Vortices

Apart from the sound mode described above, there are an interesting class of excitations known as vortices.

ϕ being a phase is only defined mod 2π.

Consider a cylindrical sample with a hole in the middle.

The field $e^{i\phi}$ must be single-valued



$$\Rightarrow \quad \oint_C \nabla\phi \cdot dl = 2\pi m$$

for every curve C that winds around the hole with m = integer.

States with different m are topologically distinct — cannot be continuously deformed into one another.

m counts the # of times the phase ϕ winds around 2π.

We can also consider such configurations directly in the Bulk where ϕ winds around by some multiple of 2π on encircling some point — these are known as vortices.

In the core of the vortex, $\langle \psi \rangle = 0$ so that the phase is not well-defined.

[Comment - such "topological" defects are a general phenomena associated with broken symmetry states. Rich possibilities depending on symmetry of order parameter space & dimensionality of coordinate space]

For a vortex with $m=1$,

$$\nabla \phi \sim \frac{m}{r} \hat{e}_\theta$$

$$\Rightarrow \quad E \sim L \int d\theta \, dr \, r \, \frac{\rho_s}{2} \left(\frac{m}{r}\right)^2$$

$$\sim \pi L \rho_s \cdot m^2 \ln \frac{L}{a}$$

a ← vortex "core" size.

Note that once the $U(1)$ symmetry is broken, we can no longer label the excitations of the system by N_{tot} (it loses it's integrity as a good quantum #).

However, there emerges a new (~~t~~opological) quantum # – the vorticity – which is conserved by the dynamics.

We may label the states in the Hilbert space by their total vorticity.

To summarize, there are two distinct classes of excitations in a neutral superfluid – the sound mode & vortices.

Sound mode is gapless with $\omega_k \sim |\vec{k}|$ & controls low-T thermodynamics.

Vortices are topologically stable objects which interact with each other thru' long ranged forces mediated by the sound modes.

(Divergence of energy of a single vortex is also due to it's interaction with the sound mode).

Superfluid equations of motion

The "hydrodynamic" variables that are necessary to describe the long length scale & long time physics of the superfluid state at zero temperature are the phase θ of the order parameter & the local particle density $\rho(x)$.

(see previous lectures)

As hinted by action on page 71, these are canonically conjugate to each other

To see this formally, consider

$$\left[\hat{\psi}(x), \psi^{\dagger}(x')\right] = \delta(x-x')$$

$$\Rightarrow \left[\hat{\psi}(x), \psi^{\dagger}(x')\psi(x')\right] = \psi(x)\,\delta(x-x')$$

For states that are only smooth long distance deformations of the superfluid ground state, write

$$\hat{\psi}(x) \approx \psi_0\, e^{i\hat{\theta}(x)} \qquad \text{so that}$$

<u>More careful discussion</u>

Actually must write $\hat{\psi}(x) = \sqrt{\rho(x)}\, e^{i\hat{\vartheta}(x)}$

with $\rho(x) = \rho_0 + \delta\rho(x)$

In the relevant portion of the Hilbert space, matrix elements of $\delta\rho(x)$ will have small absolute values compared to ρ_0.

$$\hat{\psi}^{+}(x)\,\hat{\psi}(x) = \left(e^{-i\theta(x)}\sqrt{\rho(x)}\right)\left(\sqrt{\rho(x)}\,e^{i\theta(x)}\right)$$

$$\approx \rho(x)$$

$$\left[\sqrt{\rho_0+\delta\rho}\;e^{i\theta(x)},\ \rho(x')\right] = \sqrt{\rho_0+\delta\rho}\;e^{i\vartheta}\,\delta(x-x')$$

Now approximate $\sqrt{\rho_0+\delta\rho} \approx \sqrt{\rho_0}$ in the factor in front of $e^{i\vartheta(x)}$ to get

$$\left[e^{i\theta(x)},\ \rho(x')\right] = e^{i\vartheta(x)}\,\delta(x-x')$$

$$\left[e^{i\hat{\theta}(x)}, \hat{\rho}(x') \right] = e^{i\hat{\theta}(x)} \delta(x-x')$$

This commutation relation is satisfied if

$$\left[\hat{\theta}(x), \hat{\rho}(x') \right] = -i\,\delta(x-x')$$

(As a rough check, consider expansion of 1st eqn above in powers of $\hat{\theta}$ & keep leading non-vanishing term in both LHS & RHS);

More formal proof: ~~Start with~~ $\cancel{\left[e^{i\theta}, \rho \right] = e^{i\theta} \delta(x-x')}$

~~Differentiate both sides w.r.t. θ~~

Start with $\left[\hat{\theta}(x), \hat{\rho}(x') \right] = -i\,\delta(x-x')$

$$\Rightarrow \text{Can write } \hat{\rho}(x') = i \frac{\delta}{\delta\hat{\theta}(x)}$$

$$\Rightarrow \left[e^{i\hat{\theta}(x)}, \hat{\rho}(x') \right] = i\left(e^{i\theta(x)} \frac{\delta}{\delta\theta(x')} - \frac{\delta e^{i\theta}}{\delta\theta(x')} \right)$$

$$= e^{i\theta(x)} \delta(x-x') \quad \text{as required}$$)

Thus $\hat{\theta}$ & $\hat{\rho}$ are conjugate to each other.

Now consider ~~the Ham~~ a generic Hamiltonian for the interacting Bose system.

Again restricting to states which are smooth long distance deformations of the superfluid ground state, we may write

$$\hat{H} = \hat{H}\left[\hat{\rho}(x), \hat{\theta}(x)\right] = \int d^d x \; \mathcal{H}\left[\hat{\rho}(x), \hat{\theta}(x)\right]$$

where $\hat{\rho}(x)$ = local density

$\hat{\theta}(x)$ = local phase of order parameter

$\mathcal{H}$ = Hamiltonian density

$$\frac{\partial \hat{\theta}(x,t)}{\partial t} = -\left\langle \frac{\delta \mathcal{H}}{\delta \rho(x)} \right\rangle = -\mu(x,t)$$

$\mu(x,t)$ = local chemical potential.

The density satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\vec{j}.$$

From the microscopic expression for the current operator

$$\vec{j} = \frac{\hat{\psi}^{+}\left(-i\vec{\nabla}\right)\hat{\psi}}{2m} + h.c$$

and letting $\hat{\psi} \simeq \psi_0 e^{i\hat{\theta}}$,

we get $\vec{j} = \frac{\psi_0^2}{m}\vec{\nabla}\theta = \frac{\rho_0}{m}\vec{\nabla}\theta$

(More generally, we should allow for some arbitrary coefficient ρ_s instead of ρ_0).

Then $\partial\rho/\partial t = -\frac{\rho_0}{2m}\nabla^2\theta$

Note that $\partial/\partial t\, \vec{\nabla}\theta = -\vec{\nabla}\mu$

$\Rightarrow$ If θ is uniform through the sample, the system cannot sustain chemical potential differences

$\Rightarrow$ $\frac{\partial\vec{j}}{\partial t} = -\frac{\rho_0}{m}\vec{\nabla}\mu$

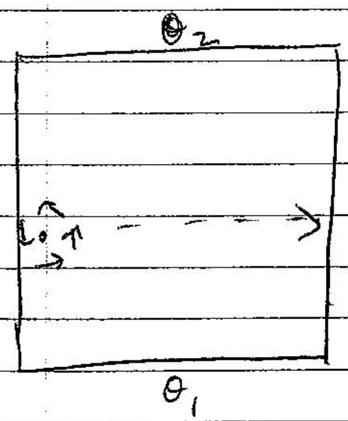
$\Rightarrow$ If there is an applied chemical potential difference, the current will increase.

In particular if $\vec{\nabla}\mu = 0$, $\dfrac{\partial \vec{j}}{\partial t} = 0$, i.e a superflow.

How can the current degrade?

This requires the motion of vortex lines.

Consider a 2-d section



Whenever a vortex moves from left to right, θ_2 winds by 2π relative to θ_1.

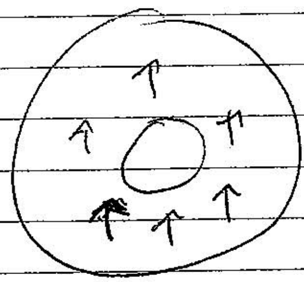
$\Rightarrow$ If $\dot{n}_v$ = rate at which vortices move from left to right,

$$\dot{\theta}_2 - \dot{\theta}_1 = 2\pi \dot{n}_v = -(\mu_2 - \mu_1)$$

$\Rightarrow$ Vortex motion can produce a chemical potential difference.

For more insight, consider a toroidal ring.

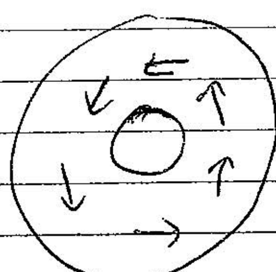
In the ground state θ = uniform



$\Rightarrow$ no current.

Now consider a state where θ winds by $2\pi n$ around the hole $\Rightarrow$ there are n vortices trapped in the hole.

($n=1$)



θ is non-uniform

$\Rightarrow$ there is a circulating current.

Degrading the current to return system to ground state requires the trapped vortices to escape from the hole.

– this has a huge energy barrier $\Rightarrow$ very improbable for a large system.

$\therefore$ Topological stability of vortex guarantees persistence of superflow

Revisit the sound (i.e Goldstone) mode from the eqns of motion:

$$\frac{\partial \phi}{\partial t} = -\mu \quad ; \quad \frac{\partial \rho}{\partial t} = -\frac{\rho_s}{m} \nabla^2 \phi .$$

$$\Rightarrow \quad \frac{\partial^2 \phi}{\partial t^2} = -\frac{\partial \mu}{\partial t} .$$

With short-ranged interactions, change in local value of μ is related to local change in density

$$\delta\mu = \frac{\delta\rho}{\kappa}$$ where κ = compressibility of the system.

(Assume no externally imposed chemical potential gradients)

$$\frac{\partial^2 \phi}{\partial t^2} = -\frac{1}{\kappa} \frac{\partial \rho}{\partial t} = \frac{\rho_s}{m\kappa} \nabla^2 \phi$$

This describes a sound wave with $\omega_k = ck$

with $c^2 = \frac{\rho_s}{m\kappa}$.

Alternately, can eliminate ϕ to get an eqn. for n:

$$\frac{\partial^2 \rho}{\partial t^2} = -\frac{\rho_s}{m} \nabla^2 \frac{\partial \phi}{\partial t} = \frac{\rho_s}{m\kappa} \nabla^2 \rho .$$

Thus the sound wave may be thought of as an oscillation in the density of the liquid.

Charged bosons

What happens if the bosons that undergo the superfluid transition are charged?

This is the case with superconductivity in metals which can be thought of as the superfluidity of Cooper pairs which are charged bosons.

The sound wave mode — being an oscillation of the charge density — will be costlier.

[Change in density in one part of the sample creates a long ranged electrostatic potential that affects the behaviour in other distant parts].

Consider the eqns of motion

$$\frac{\partial \phi}{\partial t} = -\mu, \quad \frac{\partial \rho}{\partial t} = -\frac{\rho_s}{m} \nabla^2 \phi$$

Unlike with short-ranged interactions, changes in μ are not determined by the local change in density.

To handle this situation, divide μ into two parts

$$\mu = \mu_{loc} + eV$$

where ~~μ_{loc} is the~~ V is the electrostatic potential determined by the charge density in distant parts of the sample thru' the Gauss law

$$\nabla^2 V = -4\pi e \delta\rho$$

Changes in μ_{loc} are determined by the local change in density as the effect of the long range interaction have been subtracted out.

The eqns become
$$\frac{\partial\phi}{\partial t} = -\left(\mu_{loc} + eV\right)$$

$$\frac{\partial\delta\rho}{\partial t} = -\frac{\rho_s}{m}\nabla^2\phi$$

$$\nabla^2 V = -4\pi e\delta\rho$$

$$\therefore \frac{\partial^2 \delta\rho}{\partial t^2} = -\frac{\rho_s}{m}\nabla^2 \frac{\partial\theta}{\partial t} = +\frac{\rho_s}{m}\left(\nabla^2\mu_{loc} + e\nabla^2 V\right)$$

$$= +\frac{\rho_s}{mK}\nabla^2\delta\rho - 4\pi e^2 \frac{\rho_s}{m}\delta\rho .$$

$\Rightarrow$ Dispersion $\quad \omega^2 = \frac{\rho_s k^2}{mK} + \frac{4\pi e^2}{m}\rho_s$

$\Rightarrow \quad \omega \to$ cst as $k \to 0$.

$\quad$ = ("plasma frequency").

Thus the sound mode has become gapped

– in this context, it is known as a plasmon

(for plasma oscillations).

This result illustrates an important exception to Goldstone's theorem – if there are long ranged forces, then the Goldstone mode may become gapped (the Anderson-Higgs phenomenon)

Eg: In systems where the symmetry being "broken" is a gauge symmetry (like in a SC), the Goldstone mode is gapped.

These ideas are at the heart of application of broken symmetry concepts to the standard model.

$W^{\pm}$, Z particles are the analogs of the plasmon.

Now return to the neutral superfluid

$$[\theta(x), \rho(x')] = -i\,\delta(x-x')$$

implies ~~suggests~~ that θ & ρ cannot simultaneously be specified to arbitrary accuracy.

In particular if we choose as a ground state, a state with θ uniform & fixed then it suggests that such a state will not have definite particle #

To see this explicitly, note that for a state with fixed particle # N

$$\langle \phi_N | \hat{\psi} | \phi_N \rangle = 0$$ as $\hat{\psi}$ only has non-zero matrix elements between states with different N.

$\Rightarrow$ for a closed system with N bosons, cannot have $\langle \hat{\psi} \rangle = 0$ strictly.

~~Th~~ What then are we to make of our discussion of superfluidity which assumed $\langle \hat{\psi} \rangle \neq 0$?

To see what is going on, consider a large system with N particles in a closed box of volume V.

Consider the path integral

$$Z = \int [D\psi]\, e^{iS}$$

$$S = \int dt\, d^d x \;\; \psi^* \partial_t \psi + \psi^* \frac{\nabla^2}{2m} \psi + \mu |\psi|^2 - u|\psi|^4 .$$

~~As before~~ Proceed as before: Minimize classical action to get $\psi = \psi_0 = \rho_1$ then write $\psi = \left(\sqrt{\rho_0 + \delta\rho}\right) e^{i\theta}$.

In the earlier derivation we expanded to 2nd order in $\delta\rho$ & then integrated it out to get

$$S_{eff}[\theta] = \int dt\, d^d x \;\; \frac{K}{2}(\partial_t \theta)^2 - \frac{\rho_0}{2m}(\nabla\theta)^2 .$$

To study the effects of the fluctuations

In the earlier derivation, we expanded to 2nd order in $\delta\rho$ to obtain

$$S[\delta\rho, \theta] = \int d^dx\, dt \quad -\delta\rho \frac{\partial\theta}{\partial t} - \frac{\rho_0}{2m}(\nabla\theta)^2 - \frac{1}{2K}(\delta\rho)^2$$

To study the fluctuations of the total particle # & its conjugate, consider only uniform configurations of θ, $\delta\rho$: $\theta = \theta_0$, $\delta\rho = \delta\rho_0$ independent of x.

$$S[\delta\rho_0, \theta_0] = V\int dt\left[(-\delta\rho_0)\frac{d\theta_0}{dt} - \frac{(\delta\rho_0)^2}{2K}\right]$$

Introduce $\delta N = V\delta\rho_0$ = change in total particle #

$$S[\delta N_0, \theta_0] = \int dt\left[-\delta N_0 \frac{d\theta_0}{dt} - \frac{(\delta N_0)^2}{2VK}\right]$$

$\Rightarrow$ δN_0 & θ_0 are canonically conjugate

$[\hat{\theta}_0, \hat{\delta N}_0] = -i$ in operator formulation

Can now integrate δN_0 to get

$$S_{eff}[\theta_0] = \frac{\kappa V}{2}\int dt \left(\frac{d\theta_0}{dt}\right)^2$$

which is action for QM particle on a ring of unit radius

θ_0 : angular coordinate on ring

κV ~ role of "mass".

Equivalently can write Hamiltonian

$$H_{eff} = \kappa/_{2V}(\delta N)^2 = \kappa\frac{(N-N_0)^2}{2V}$$

Key point: "Mass" of particle $\propto V \to \infty$ in thermodynamic limit $\Rightarrow$ Can treat particle "classically"

$\Rightarrow$ Can ignore non-commutivity of θ_0 & δN_0

$\Rightarrow$ Can assign simultaneous eigenvalues to θ_0 & δN_0.

Alternately if N_0 is particle # in gd state, there exist excited states with $N - N_0 \sim O(1)$ with energy of order $\frac{1}{V}$

$\Rightarrow$ In the thermodynamic limit all these states become degenerate

(Note that sound wave states have energy $\sim O(1/L)$ $\gg$ '$1/V$' for $d > 1$).

Equivalently for a large system if we initially prepare the system in a "wave-packet" with reasonally well defined θ_0, ~~dr with~~ the uncertainty in θ_0 will evolve very slowly.