8.513 Problem Set # 1

Problems:

1. (10 pts.) **Energy conservation and Poisson bracket**

   Consider a system described by the generalized equation of motion $\hbar b_{ij}(\xi^i)\dot{\xi}^j = \frac{\partial H}{\partial \xi^i}$, where $b_{ij} = \partial_{\xi^i} a_j - \partial_{\xi^j} a_i$.

   (a) Show that the time dependence of a quantity $O(\xi^i)$ is given by

   $$\hbar \frac{\mathrm{d}}{\mathrm{d}t} O = [O, H]_{\mathrm{cl}}, \qquad [A, B]_{\mathrm{cl}} \equiv b^{ij} \frac{\partial A}{\partial \xi^i} \frac{\partial B}{\partial \xi^j} \qquad \text{(generalized Poisson bracket)}$$

   where b^{ij} is the inverse of b_{ij}: $b^{ik} b_{kj} = \delta^i_j$.
   (b) Show that, just like the quantum commutator $[A, B] = -[A, B]$, $[A, B]_{\mathrm{cl}} = -[B, A]_{\mathrm{cl}}$.
   (c) Show that the energy is conserved $\mathrm{d}H/\mathrm{d}t = 0$.

   (Replacing Poisson bracket by quantum commutator allows us to quantize a classical theory)

2. (10 pts.) **Effective quantum Hamiltonian of a particle in the first Landau level**

   For a charge-1 particle in a 2D plane with a strong uniform magnetic field B, the motion of the average position (*ie* the center of the cyclotron orbit) of the particle is described by equations of motion

   $$\dot{x}^i = B^{-1} \epsilon^{ij} \frac{\partial V}{\partial x^j}, \qquad i, j = 1, 2$$

   where $V(x^i)$ is the potential energy (the Hamiltonian) of the particle. Here $\epsilon^{12} = -\epsilon^{21} = 1$ and $\epsilon^{11} = \epsilon^{22} = 0$.

   (The above equations of motion can all be viewed as the classical equations of motion for a mass m particle magnetic field, but in the $m \to 0$ limit. In this limit, the coordinate-space classical equations of motion become first order in time derivative. *ie* the coordinate-space equations of motion becomes a phase-space equations of motion in this limit. In this problem, we also see that knowing phase-space equations of motion and phase-space energy function allows us to obtain phase-space Lagrangian.)

   (a) Find the phase space Lagrangian that reproduces the above equations of motion with $V(x^i)$ as the energy (*ie* the Hamiltonian).

   (b) Find the quantum description of the above classical system. (*ie* find the quantum Hamiltonian operator).

   (c) Assume $V(x^i) = \frac{v}{2}[(x^1)^2 + (x^2)^2]$. Find the energy levels of the quantum system obtained in (b).

   (Such a quantum description describes the particle in the first Landau level.)

3. (10 pts) **Differential form**

   (a) Let $(a_x(x, y), a_y(x, y))$ be a vector field in 2-dim space (x, y). Let $a = a_x \mathrm{d}x + a_y \mathrm{d}y$ be the corresponding differential 1-form. Let $b = \mathrm{d}a = b_{xy} \mathrm{d}x\mathrm{d}y$. Show that

   $$\int_{D^2} b = \int_{D^2} b_{xy} \mathrm{d}x\mathrm{d}y = \oint_{\partial D^2} a_x \mathrm{d}x + a_y \mathrm{d}y \qquad (1)$$

   where ∂D^2 is the boundary of a disk D^2 in the 2-dim space.

(b) Let $S^2(r)$ be a sphere in 3-dim space $\mathbb{R}^3 = (x, y, z)$ of radius r. Let $b = r^{-3}(x\mathrm{d}y \wedge \mathrm{d}z - y\mathrm{d}x \wedge \mathrm{d}z + z\mathrm{d}x \wedge \mathrm{d}y)$ be a 2-form. Show that $\mathrm{d}b = 0$ (*ie* b is a closed 2-form) away from the origin $(x, y, z) = (0, 0, 0)$. Calculate $\int_{S^2(r)} b$, which should be independent of r. (Hint: for $x = r\sin(\theta)\cos(\phi)$, $y = r\sin(\theta)\sin(\phi)$, $z = r\cos(\theta)$, $\mathrm{d}z = \cos(\theta)\mathrm{d}r - \sin(\theta)\mathrm{d}\theta$, etc.)

4. (10 pts) **Semi-classical picture of a quantum spin**

   Consider a quantum spin-S system described by

$$\hat{H} = -\boldsymbol{B} \cdot \hat{\boldsymbol{S}}, \qquad \boldsymbol{B} = B\boldsymbol{z}.$$

   We like to use the coherent-state approach to obtain the corresponding classical system of the above quantum spin model. We choose $|\boldsymbol{n}\rangle$ to be our coherent states, where $|\boldsymbol{n}\rangle$ is the eigenstate of $\boldsymbol{n} \cdot \hat{\boldsymbol{S}}$ with the maximal eigenvalue S.

   (a) Find the phase-space Lagrangian $L(\boldsymbol{n}, \dot{\boldsymbol{n}})$ that describes the classical motion of the unit vector $\boldsymbol{n}$.

   (b) Find the classical equation of motion for $\boldsymbol{n}$.
   (Hint: you may assume $S = 1/2$ first and guess the results for higher spins.)

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