8.513 Problem Set # 10

Problems:

1. (20 pts) **1D p-wave superconductor on an open chain** 20

   We have seen that an Ising model on an open chain $i = 1, 2, \cdots, L$

   $$H = -\sum_{i=1}^{L} (J\sigma_i^x \sigma_{i+1}^x + h\sigma_i^z)$$

   can be described by the following Majorana fermion model

   $$H = \sum_{i=1}^{L-1} \mathrm{i}J\lambda_i^y \lambda_{i+1}^x + \sum_{i=1}^{L} \mathrm{i}h\lambda_i^x \lambda_i^y$$

   Such a fermion model is in fact a model for 1D p-wave superconductor for a complex fermion

   $$c_i = \frac{1}{2}(\lambda_i^x + \mathrm{i}\lambda_i^y) \tag{1}$$

   (a) Assume $J = 0$. Show that many-body ground state of the above p-wave superconductor is not degenerate.
   (Hint: you may write H in terms of complex fermion c_i's).

   (b) Assume $h = 0$. Show that many-body ground state of the above p-wave superconductor has a two-fold degeneracy.
   (Hint: you may introduce complex fermion operators $\tilde{c}_i = \lambda_i^y + \mathrm{i}\lambda_{i+1}^x$ and $\tilde{c}_0 = \lambda_L^y + \mathrm{i}\lambda_1^x$, and write H in terms of those.).

   (c) Find the single particle energy levels of Majorana fermions for general J, h. Consider the two limits discussed above.
   (Hint: The following wave functions (not normalized) $\sin(\frac{n\pi}{L+1}i)$, $n = 1, 2, \cdots, L$, may be helpful. Note that $\sin(\frac{n\pi}{L+1}i) = 0$ for $i = 0$ and $i = L+1$. This enforces the boundary conditions for fermions: their amplitude at $i = 0$ and $i = L+1$ vanishes.)

   So there is a phase transition within 1D p-wave superconductor as we change h/J. The phase with two-fold degeneracy is a topological p-wave superconductor. The phase with no degeneracy is a trivial p-wave superconductor.

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