8.513 Problem Set # 11

Problems:

1. (20 pts) **Realizing and classifying 3d topological insulator using Dirac fermions**

   A 3d topological insulator has a symmetry $U^f_{\text{charge}}(1) \rtimes Z_4^T/Z_2^f$ (*ie* charge conservation and time reversal symmetry). To obtain a realization of such a topological insulator, we may first construct an one-body gapless Dirac fermion Hamiltonian in real bases, then we add a proper mass term to realizing a 3d topological insulator.

   (a) Consider a 3d non-interacting fermion system, described by a $\boldsymbol{k}$-space one-body Hamiltonian

   $$M(\boldsymbol{k}) = \boldsymbol{v}(\boldsymbol{k}) \cdot \boldsymbol{\sigma}, \tag{1}$$

   where $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the three Pauli matrices, and $\boldsymbol{v}(\boldsymbol{k})$ is a smooth vector fields in the Brillouin Zone. Assume that $\boldsymbol{v}(\boldsymbol{k})$ has only two zeros at $\boldsymbol{k}_R$ and $\boldsymbol{k}_L$:

   $$M(\boldsymbol{k}_R + \delta\boldsymbol{k}) = \delta k_x \sigma^1 + \delta k_y \sigma^2 + \delta k_z \sigma^3, \quad M(\boldsymbol{k}_L + \delta\boldsymbol{k}) = -\delta k_x \sigma^1 - \delta k_y \sigma^2 - \delta k_z \sigma^3. \tag{2}$$

   Show that the one-body Hamiltonian in the continuum $\boldsymbol{x}$-space is given by

   $$H_{\text{one-body}} = \sigma^1 \otimes \sigma^3 \mathrm{i}\partial_x + \sigma^2 \otimes \sigma^3 \mathrm{i}\partial_y + \sigma^3 \otimes \sigma^3 \mathrm{i}\partial_z. \tag{3}$$

   The corresponding many-body Hamiltonian is given by

   $$\hat{H} = \int \mathrm{d}\boldsymbol{x}^3 \; \hat{c}_b^\dagger (H_{\text{one-body}})_{ba} \hat{c}_a \tag{4}$$

   (b) We like to rewrite the many-body Hamiltonian in terms of Majorana fermions

   $$\hat{H} = \int \mathrm{d}\boldsymbol{x}^3 \; \hat{\eta}_\beta (H^R_{\text{one-body}})_{\beta\alpha} \hat{\eta}_\alpha. \tag{5}$$

   The corresponding one-body Majorana-fermions Hamiltonian can by obtained by replacing $-\mathrm{i}$ by $\varepsilon = \mathrm{i}\sigma^2$ (see page 14 of Lecture note 3). We first needs to make i explicit (this step is crucial)

   $$\begin{aligned} H_{\text{one-body}} &= \mathrm{i}\sigma^1 \otimes \sigma^3 \partial_x + \varepsilon \otimes \sigma^3 \partial_y + \mathrm{i}\sigma^3 \otimes \sigma^3 \partial_z \\ &= \mathrm{i}(\sigma^1 \otimes \sigma^3 \partial_x - \mathrm{i}\varepsilon \otimes \sigma^3 \partial_y + \sigma^3 \otimes \sigma^3 \partial_z). \end{aligned} \tag{6}$$

   Then, inside the bracket, we replace $-\mathrm{i}$ by $\varepsilon = \mathrm{i}\sigma^2$ and 1 by σ^0 to obtain

   $$H^R_{\text{one-body}} = \mathrm{i}(\sigma^0 \otimes \sigma^1 \otimes \sigma^3 \partial_x + \varepsilon \otimes \varepsilon \otimes \sigma^3 \partial_y + \sigma^0 \otimes \sigma^3 \otimes \sigma^3 \partial_z) \tag{7}$$

   The 8×8 gamma matrices

   $$\gamma_1 = \sigma^0 \otimes \sigma^1 \otimes \sigma^3, \quad \gamma_2 = \varepsilon \otimes \varepsilon \otimes \sigma^3, \quad \gamma_3 = \sigma^0 \otimes \sigma^3 \otimes \sigma^3, \tag{8}$$

   are real symmetric and satisfy $\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij}$. Just like the 2d case discussed in the class, the $U^f(1)$ symmetry and time reversal symmetry generators are given by

   $$Q = \varepsilon \otimes \sigma^0 \otimes \sigma^0, \quad T = \sigma^3 \otimes \varepsilon \otimes \sigma^0, \quad TQ = \sigma^1 \otimes \varepsilon \otimes \sigma^0, \tag{9}$$

which satisfy

$$QT = -TQ, \quad \gamma_i T = -T\gamma_i, \quad \gamma_i Q = Q\gamma_i, \quad \gamma_i TQ = -TQ\gamma_i. \tag{10}$$

The mass term A is a real anti-symmetric matrix satisfying

$$A\gamma_i = -\gamma_i A, \quad AT = -TA, \quad AQ = QA, \quad ATQ = -TQA, \tag{11}$$

Find the most general form of the mass term.

(c) Show that the space of the mass matrix A is $\mathcal{R}_3^2 = \mathcal{R}_{2-3+2} = \mathcal{R}_1$. We also have $\mathcal{R}_3^2 = \mathcal{R}_1^0$.

(d) Let us calculate the space $\mathcal{R}_1^0$, which is the space of $2n \times 2n$ anti-symmetric real matrices A that satisfy

$$A^2 = -I_{2n}, \quad A\rho = -\rho A \tag{12}$$

where ρ is a real orthogonal matrix that satisfies

$$\rho^2 = 1. \tag{13}$$

We may choose a basis such that

$$\rho = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix} \tag{14}$$

where I_n is the $n \times n$ identity matrix.

Show that A has a form

$$A = \begin{pmatrix} 0 & O_n \\ -O_n^\top & 0 \end{pmatrix}, \qquad O_n \in O(n). \tag{15}$$

Thus $\mathcal{R}_1^0 = O(n)$ and $\pi_0(\mathcal{R}_1^0) = \mathbb{Z}_2$. This suggests that in 3+1D, the gapped non-interacting fermions with $U^f_{\text{charge}}(1) \rtimes Z_4^T/Z_2^f$ symmetry has only two phases, a trivial phase and a topological insulator phase.

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