8.513 Problem Set # 2

Problems:

1. (15 pts) **A particle on torus with uniform magnetic field**

   Consider a particle on a torus of size $L_x \times L_y$, described by $\mathrm{i}\partial_t\psi(x,y,t) = H\psi(x,y,t)$, $H = -\frac{m}{2}\boldsymbol{\partial}^2$. (In this class we often use $\hbar = e = c = 1$ unit.) The wave function satisfy the periodic boundary condition

$$\psi(x+L_x, y) = \psi(x, y+L_y) = \psi(x,y). \tag{1}$$

   Now consider a charge-1 particle on a torus of size $L_x \times L_y$ with a uniform magnetic field B, described by $\mathrm{i}\partial_t\psi(x,y,t) = H\partial_t\psi(x,y,t)$, $H = -\frac{m}{2}(\boldsymbol{\partial} - \mathrm{i}\boldsymbol{A})^2$.

   (a) Under the gauge transformation $\psi \to \psi' = \mathrm{e}^{\mathrm{i}f(x,y)}\psi$, the equation of motion is changed to $\mathrm{i}\partial_t\psi' = H'\psi'$. Find the gauge transformed H'.

   (b) A uniform magnetic field B can be described by $(A_x, A_y) = (-By, 0)$. We note that $\boldsymbol{A}(x,y) = \boldsymbol{A}(x+L_x, y)$ and the wave function can satisfy the periodic boundary condition in the x-direction

$$\psi(x+L_x, y) = \psi(x,y). \tag{2}$$

   But $\boldsymbol{A}(x,y) \neq \boldsymbol{A}(x, y+L_y)$ and the wave function does not satisfy the periodic boundary condition in the y-direction. But $\boldsymbol{A}(x,y)$ and $\boldsymbol{A}(x, y+L_y)$ only differ by a $U(1)$-gauge transformation:

$$\boldsymbol{A}(x, y+L_y) = \boldsymbol{A}(x,y) - \mathrm{i}U^{-1}\boldsymbol{\partial}U, \quad U = \mathrm{e}^{\mathrm{i}f}. \tag{3}$$

   Find $U(x,y)$. Find the modified "periodic" boundary condition in the y-direction

$$\psi(x, y+L_y) = ?\psi(x,y). \tag{4}$$

   Show that, in order for the modified "periodic" boundary condition to be consistent, the magnetic field on torus must be quantized, such that total number of flux quanta (the Chern number) through the torus must be an integer.

   (c) Show that the number degenerate states in the first Landau level is equal to the number of flux quanta.

   We have seen that the motion of a particle in the first Landau level is described by a phase space which is nothing but the torus (x,y). In this case, the number of flux quantum of the phase-space magnetic field is equal to the total number of states or

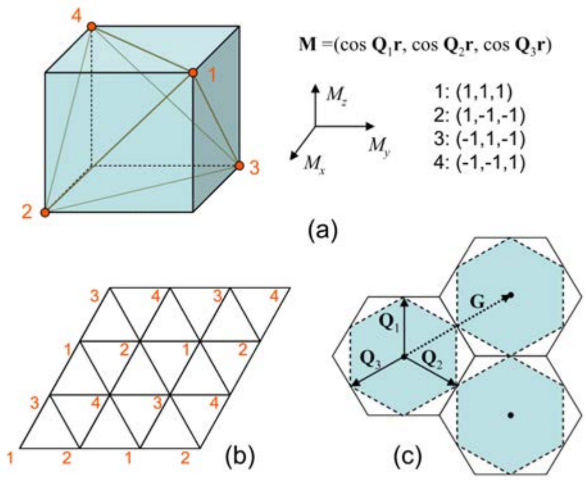
$$\text{Chern number of phase-space magnetic field} = \text{number of states.} \tag{5}$$

   on torus. But for spin-S, the phase space is a sphere, which lead to a different result

$$\text{Chern number of phase-space magnetic field} = \text{number of states} - 1. \tag{6}$$

2. (20 pts) **Electron hopping in a 2D chiral magnet of triangular lattice**

We consider electron hopping on a triangular lattice. The lattice has a non coplanar magnetic order, where the magnetic moments may point in one of the for directions, $\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3, \boldsymbol{n}_4$, as shown in the figure below.



Such a magnetic order is called chiral magnetic order due to its handness.

Due to magnetic interaction, we assume the electron spin state

$$|\boldsymbol{n}_i\rangle = \begin{pmatrix} \cos(\theta_i/2) \\ e^{i\varphi_i}\sin(\theta_i/2) \end{pmatrix} \tag{7}$$

on each site to be always parallel to the magnetic moment on that site. The electron hopping amplitude t_{ij} from site-i to site-j is given by overlap of the electron spin wave function

$$t_{ij} = A\langle \boldsymbol{n}_i|\boldsymbol{n}_j\rangle, \qquad A > 0. \tag{8}$$

In general, t_{ij} is complex, and the total phase of hopping around a triangle ijk is the phase of the product of three hoppings: $t_{ij}t_{jk}t_{ki}$. Such a phase correspond to magnetic flux through the triangle ijk.

(a) Show that the flux is uniform and compute the value of the flux for each triangle. (We see that the chiral magnetic order can simulate a uniform magnetic field.)

(b) Show that t_{ij}'s do not have the translation symmetry of the triangular lattice. Show that we can re-adjust the phases of the spin states $|\boldsymbol{n}_1\rangle$, $|\boldsymbol{n}_2\rangle$, $|\boldsymbol{n}_3\rangle$, $|\boldsymbol{n}_4\rangle$, such that t_{ij}'s only double the unit cell of the triangular lattice.

(c) Since the doubled unit cell contains two sites, there are two bands. Find the dispersions of the two bands. (Hint: You may view the triangular lattice as a square lattice with a diagonal link.) What is the gap between the two bands?

(d) Now assume the hopping on the horizontal links are zero. Find the new dispersions of the two bands. What is the gap between the two bands?

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