### 8.513 Problem Set \# 5

## Problems:

## 1. (20 pts) New critical point of superfluid transition

We map a 1D interacting boson system to spin-1 chain described by

$$
\begin{equation*}
H=\sum_{i}\left[-J\left(S_{i}^{x} S_{i+1}^{x}+S_{i}^{y} S_{i+1}^{y}\right)-B S_{i}^{z}+V\left(S_{i}^{z}\right)^{2}\right], \tag{1}
\end{equation*}
$$

where

$$
S_{x}=\frac{1}{\sqrt{2}}\left(\begin{array}{lll}
0 & 1 & 0  \tag{2}\\
1 & 0 & 1 \\
0 & 1 & 0
\end{array}\right), \quad S_{y}=\frac{\mathrm{i}}{\sqrt{2}}\left(\begin{array}{ccc}
0 & -1 & 0 \\
1 & 0 & -1 \\
0 & 1 & 0
\end{array}\right), \quad S_{z}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{array}\right)
$$

The interacting boson or spin-1 chain has the following phase diagram


In the above, the black lines mark the critical point of superfluid transition that we studied in the last homework, where the gapless mode has a quadratic dispersion $\omega_{k} \propto k^{2}$. The filled dot represents a different critical point of superfluid transition where the gapless mode has a linear dispersion $\omega_{k} \propto|k|$. In this problem, we like to show this feature.
(a) Consider a many-body trial state $|\Psi\rangle=\otimes_{i}\left|\phi_{i}, \tilde{\phi}_{i}\right\rangle$, where spin- 1 state on site- $i$ is given by

$$
\left|\phi_{i}, \tilde{\phi}_{i}\right\rangle=\frac{1}{\sqrt{1+\left|\phi_{i}\right|^{2}+\left|\tilde{\phi}_{i}\right|^{2}}}\left(\begin{array}{c}
\phi_{i}  \tag{3}\\
1 \\
\tilde{\phi}_{i}
\end{array}\right)
$$

When $V$ is large, the ground state is a Mott insulator given by $\phi_{i}=\tilde{\phi}_{i}=0$. Find the phase space Lagrangian that describe the semi classical dynamics of $\phi_{i}(t), \tilde{\phi}_{i}(t)$ (keep only the quadratic terms.)
(b) Assume $B=0$. Start from a large $V$ Mott insulating phase. Reduce $V$ to reach the superfluid phase transition. Find the low energy dispersion relation of the gapless mode at the transition. What is the low energy dispersion relation of the gapless mode at the transition if $B \neq 0$ ? You should see the $z=1$ and $z=2$ critical points for the two cases.
(A similar result hold for higher dimensions.)
2. ( 15 pts ) The equivalence between 1D bosons and 1D fermions

Consider 1D right moving fermions with dispersion relation $\omega(k)=v k, k=\frac{2 \pi}{L} \times$ integers. Assume that the ground state has $k \leq 0$ orbitals filled.
(a) Find the many-body spectrum of the low energy excitations that have the same fermion number as the ground state. (ie find the degeneracies, say, for the first 5 many- body energy levels.) Compare the obtained many-body spectrum with the many-body spectrum of 1D right moving boson with the same dispersion relation $\omega(k)=v k$, where $k=\frac{2 \pi}{L} \times$ positive non-zero integers.
(b) Compute the heat capacity of the 1D right moving fermions, and compare it with the heat capacity of 1D right moving boson with the same velocity $v$.

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