8.513 Problem Set # 6

Problems:

1. (30 pts) **Solve XY model by mapping it to a free fermion model**

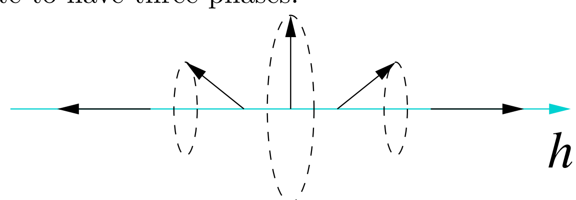
   Consider a 1+1D XY model on an open change of N sites:

$$H = \sum_{1}^{N-1} [-(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y) - h\sigma_i^z] \tag{1}$$

   Note that model has a $U(1)$ symmetry generated by σ^z spin rotation

$$U(\theta) = \prod_i \mathrm{e}^{\mathrm{i}\frac{\theta}{2}\sigma_i^z}. \tag{2}$$

   We expect the ground state to have three phases:



   When $h = 0$, the spin in the ground state all point in the same direction in the σ^x-σ^y plane. When $h \neq 0$, the spins develop an σ^z component. (The above is a classical picture which is not exactly correct.) When $h > h_c$ or $h < -h_c$, the spins all point to σ^z or $-\sigma^z$ direction. There are two phase transitions at $|h| = h_c$.

   (a) Use the Jordan-Wigner transformation to map the above the XY model to a non-interacting fermion model.

   (b) Now we pretend the fermion model to be the one on a ring of $N =$ even sites, so that the fermion model has the translation symmetry. This will simplify the calculation. Find the ground state energy $E_{\text{ring}}(h, N)$ as a function of h and N.

   (c) Find the transition point $\pm h_c$.

   (d) Show that in the small $|h|$ phase, we have emergence of Lorentz invariance, *ie* the low energy mode has a linear dispersion $\epsilon_k = v|k|$. What is the central charge of this gapless phase? The ground state energy $E_{\text{ring}}(h, N)$ has a linear N term, constant N^0 term, and possible N^{-1} term, *etc* . Write the constant N^{-1} term in term of velocity v. (The N^{-1} term also reveals the central charge of the gapless phase.)

   (e) What is the behavior of the velocity v as we approach the transition point $|h| = h_c$? What is the dispersion relation for the low energy mode at the critical point $|h| = h_c$?

   (f) In the neutron scattering experiment for the $h = 0$ state, we assume neutron dump a momentum k to the ground state $|\Psi_0\rangle$. We also assume the neutron dump a spin $S_z = 1$ to the ground state $|\Psi_0\rangle$ (*ie* the neutron has a spin flip $\Delta S_z = 1$). Such as scattering creates an excited state $|\Psi_k\rangle = [\sum_i \mathrm{e}^{\mathrm{i}ki}\sigma_i^-]|\Psi_0\rangle$ (σ_i^- changes S_z by 1). We know that $|\Psi_k^f\rangle = [\sum_i \mathrm{e}^{\mathrm{i}ki}c_i]|\Psi_0\rangle$ is an energy eigenstate. Is $|\Psi_k\rangle = [\sum_i \mathrm{e}^{\mathrm{i}ki}\sigma_i^-]|\Psi_0\rangle$ an energy eigenstate? In the above neutron scattering experiment for the $h = 0$ state, can we observe a δ-function spectrum that following the fermion c_k dispersion relation, or we observe a continuum spectrum? Try to describe the neutron scattering spectrum.

2. (10 pts) **An 1D fermion mass term may not give fermions a mass gap**

Consider an 1D model with two chiral fermions described by the following Hamiltonian on a ring

$$H = \int_0^L \mathrm{d}x \; c_1^\dagger(x) v_1 \mathrm{i}\partial_x c_1(x) + c_2^\dagger(x) v_2 \mathrm{i}\partial_x c_2(x) + M(c_1^\dagger(x) c_2(x) + h.c.) \tag{3}$$

where $M(c_1^\dagger(x)c_2(x) + h.c.)$ is the mass term. In the continuum, the fermion operators satisfy

$$\{c_i(x), c_j(y)\} = \{c_i^\dagger(x), c_j^\dagger(y)\} = 0, \qquad \{c_i(x), c_j^\dagger(y)\} = \delta(x-y)\delta_{ij}, \tag{4}$$

Find the single particle spectrum of the above model. (Hint: you may want to go to k-space first $\psi_i^\dagger(k) = \int_0^L \mathrm{d}x \; \mathrm{e}^{\mathrm{i}kx} c_i^\dagger(x)$), for both cases $v_1 v_2 > 0$ (chiral central charge $c \equiv c_R - c_L = \pm 2$) and $v_1 v_2 < 0$ (chiral central charge $c \equiv c_R - c_L = 0$).

The mass term corresponds to a relevant perturbation, you will see that when $v_1 v_2 < 0$, the mass term make the system to be gapped. However, when $v_1 v_2 > 0$, the system remain gapless even in the presence of the mass term. The 1D system can be gapped only when chiral central charge $c \equiv c_R - c_L = 0$.

8.513 Modern Quantum Many-body Physics for Condensed Matter Systems
Fall 2021