8.513 Problem Set # 7

Problems:

1. (5 pts) **Density of many-body states and heat capacity**

   Consider an 1D chiral bosons with a linear dispersion $\epsilon_k = \hbar v k$ on a ring of length L. The degeneracy of many-body states at energy $n\hbar v\frac{2\pi}{L}$ is given the partition number p_n. For large n, we have

$$p_n \sim \frac{1}{4n\sqrt{3}} \mathrm{e}^{2\pi\sqrt{n/6}} \tag{1}$$

   (https://www.theoremoftheday.org/NumberTheory/Partitions/TotDPartitions.pdf)
   So the density of many-body states for chiral bosons is given by

$$D_{\text{many}}(E) = \frac{p_n}{v\frac{2\pi}{L}}\Big|_{E=n\hbar v\frac{2\pi}{L}} = \frac{1}{4\sqrt{3}E} \mathrm{e}^{2\pi\sqrt{EL/12\hbar v\pi}} \tag{2}$$

   ($D_{\text{many}}(E)\Delta E$ is the number of many-body states with energy in the window $[E, E+\Delta E]$.)

   (a) Use the above expression of the density of many-body states for chiral bosons to calculate the heat capacity of the system. Hint: for large L (*ie* $k_B T \gg \hbar v\frac{2\pi}{L}$), one can use saddle-point approximation to evaluate the integral
   (https://physics.stackexchange.com/questions/14639)
   Use such a result to show that for a 1D theory with central charge c and velocity v, its density of many-body states on a ring of length L has a form

$$D_{\text{many}}(E) \sim \mathrm{e}^{cL\sqrt{\varepsilon\pi/3\hbar v}}, \quad \varepsilon \equiv \frac{E}{L} = \text{energy density} \tag{3}$$

   This is how the central charge c is related to the density of many-body states.

2. (30 pts) **Jordan-Wigner transformation on a ring**:

   (a) Consider a transverse Ising model on a ring of size L (assume $L = 0$ mod 4 to be safe):

$$H = -\sum_{i=1}^{L} (J\sigma_i^x \sigma_{i+1}^x + h\sigma_i^z)$$

   where $\sigma_{L+1}^l = \sigma_1^l$. Use the Jordan-Wigner transformation

$$\lambda_i^x = (\prod_{j<i} \sigma_j^z)\sigma_i^x, \quad \lambda_i^y = (\prod_{j<i} \sigma_j^z)\sigma_i^y,$$

   to map the above spin model to a fermion model on the ring, where λ_i^x, λ_i^y are Majorana operators.

   (b) Show that the spin model and the following two fermion models on a ring

$$H^{\text{perio}} = \sum_{i=1}^{L} J\mathrm{i}\lambda_i^y \lambda_{i+1}^x + h\mathrm{i}\lambda_i^x \lambda_i^y, \quad \lambda_{L+1}^x = \lambda_1^x,\ \lambda_{L+1}^y = \lambda_1^y,$$

$$H^{\text{anti-perio}} = \sum_{i=1}^{L} J\mathrm{i}\lambda_i^y \lambda_{i+1}^x + h\mathrm{i}\lambda_i^x \lambda_i^y, \quad \lambda_{L+1}^x = -\lambda_1^x,\ \lambda_{L+1}^y = -\lambda_1^y,$$

are closely related. The states in the spin model belong to two sectors: the even sector with $Z = \prod_i \sigma_i^z = +1$ and the odd sector with $Z = \prod_i \sigma_i^z = -1$. The states in the fermion model also belong to two sectors: the even sector with $FNP = \prod_i (\mathrm{i}\lambda_i^y \lambda_i^x) = +1$ and the odd sector with $FNP = \prod_i (\mathrm{i}\lambda_i^y \lambda_i^x) = -1$. Show that the eigenstates of H^{perio} in the even (or maybe odd) sector happen to be the eigenstates of the spin model in the even sector with the same energies. The eigenstates of $H^{\text{anti-perio}}$ in the odd (or maybe even) sector happen to be the eigenstates of the spin model in the odd sector with the same energies.

(c) Find FNP for the lowest energy state in the sectors of periodic or anti-periodic boundary conditions.

(Hint: Write the fermion model in terms of the fermion operator in k-space. Note that the fermion operators with $(k, -k)$ form a subsystem that decouples from other subsystems. In fact, in k-space $(\lambda_k^x)^\dagger = \lambda_{-k}^x$. What is the FNP for such a subsystem? You need to express $FNP = \prod_i (\mathrm{i}\lambda_i^y \lambda_i^x) = \mathrm{e}^{\mathrm{i}\pi \sum_i (1-\mathrm{i}\lambda_i^y \lambda_i^x)/2}$ in k-space. You also need to consider $k = 0$ and $k = \pi$ separately, if they are present.)

(d) In the symmetry breaking phase $J > |h|$ of the Ising model, show the energy splitting Δ of the two lowest energy states to have a form

$$\Delta = O(L^\mu)\mathrm{e}^{-L/\xi}, \tag{4}$$

and find an expression of the correlation length ξ in terms of J, h.
(Hint: $\sqrt{A - B\cos(k)}$ $(A, B > 0,\ A > B)$ has branch cuts in the complex k-plane starting at $k = \pm\mathrm{i}\kappa + 2\pi n$ where κ is determined by $A = B\frac{\mathrm{e}^\kappa + \mathrm{e}^{-\kappa}}{2}$. See the attached paper, mainly eqn 4.1 – eqn. 4.5.)

3. (10 pts) **Time-ordered correlation in imaginary time**

Non-equal-time time-ordered correlation in imaginary time for fermionic operator is defined as

$$\langle T[c(x_1, t_1)c^\dagger(x_2, t_2)]\rangle = \begin{cases} \langle 0|c(x_1)\mathrm{e}^{-|t_1-t_2|H}c^\dagger(x_2)|0\rangle, & t_1 - t_2 \geq 0 \\ -\langle 0|c^\dagger(x_2)\mathrm{e}^{-|t_1-t_2|H}c(x_1)|0\rangle, & t_1 - t_2 < 0 \end{cases} \tag{5}$$

The time-ordered correlation in imaginary time for bosonic operator is defined as

$$\langle T[b(x_1, t_1)b^\dagger(x_2, t_2)]\rangle = \begin{cases} \langle 0|b(x_1)\mathrm{e}^{-|t_1-t_2|H}b^\dagger(x_2)|0\rangle, & t_1 - t_2 \geq 0 \\ \langle 0|b^\dagger(x_2)\mathrm{e}^{-|t_1-t_2|H}b(x_1)|0\rangle, & t_1 - t_2 < 0 \end{cases} \tag{6}$$

A 1D **chiral Majorana fermion** system on a ring of length L is described by the Hamiltonian in k-space:

$$H = \sum_{k>0} vk\lambda_k^\dagger \lambda_k, \quad \{\lambda_{k_1}, \lambda_{k_2}\} = 0, \quad \{\lambda_{k_1}^\dagger, \lambda_{k_2}\} = \delta_{k_1,k_2}, \tag{7}$$

where $k = \frac{2\pi}{L}(\text{integer} + \frac{1}{2})$ (assuming the fermion has the anti-periodic boundary condition). The ground state $|0\rangle$ satisfies $\lambda_k|0\rangle = 0$, $k > 0$. Such a system is a central charge $c = 1/2$ conformal field theory (CFT).

(a) We define $\lambda_{-k} = \lambda_k^\dagger$. Show that

$$\langle T[\lambda_{k_1}(t)\lambda_{k_2}(0)]\rangle = \begin{cases} \frac{1-\mathrm{sgn}(k_2)}{2}\mathrm{e}^{-v|tk_2|}\delta_{k_1+k_2}, & t \geq 0; \\ -\frac{1+\mathrm{sgn}(k_2)}{2}\mathrm{e}^{-v|tk_2|}\delta_{k_1+k_2}, & t < 0. \end{cases} \tag{8}$$

(b) Define the Majorana fermion field in real space as

$$\lambda(x) = \sum_k L^{-1/2}\lambda_k \mathrm{e}^{\mathrm{i}kx}, \quad k = \frac{2\pi}{L}(\text{integer} + \frac{1}{2}) \tag{9}$$

Show that $\lambda(x) = \lambda^\dagger(x)$ and $\{\lambda(x), \lambda(y)\} = 0$ if $x \neq y$. *ie* the fermion field is *local*. (In fact $\{\lambda(x), \lambda(y)\} = \delta(x - y)$.)

(c) Show that the time-ordered correlation for Majorana fermion field $\lambda(x, t)$ is

$$\langle \lambda(x, t)\lambda(0, 0)\rangle = \frac{L^{-1}}{\mathrm{e}^{-\mathrm{i}\pi(x+\mathrm{i}vt)/L} - \mathrm{e}^{\mathrm{i}\pi(x+\mathrm{i}vt)/L}} = \frac{(-2\pi\mathrm{i})^{-1}}{x + \mathrm{i}vt}\ _{L\to\infty} \tag{10}$$