8.513 Problem Set # 8

Problems:

1. (15 pts) $U(1)$ **non-linear σ-model field theory ("coordinate"-space Lagrangian)**

   In the class, we show that a 1d superfluid, at low energies, can be described by the following $U(1)$ non-linear σ-model (the "coordinate"-space Lagrangian)

$$L = \int \mathrm{d}x \; \frac{V_2^{-1}}{4\pi}(\partial_t\theta)^2 - \frac{V_1}{4\pi}(\partial_x\theta)^2 \; \underbrace{+\frac{\bar\phi^2}{a}\partial_t\theta}_{\text{a topo. term}}$$
$$= \int \mathrm{d}x \; \frac{V_2^{-1}}{4\pi}(\mathrm{i}u^*\partial_t u)^2 - \frac{V_1}{4\pi}(\mathrm{i}u^*\partial_x u)^2 - \mathrm{i}\frac{\bar\phi^2}{a}u^*\partial_t u, \tag{1}$$

   where $u(x) = \mathrm{e}^{\mathrm{i}\theta(x)}$. We have seen that $k \neq 0$ modes describe the low energy excitations with momentum $K \sim 0$ (the phonons).

   (a) The $k = 0$ mode can give rise to other sectors. The $k = 0$ mode is described by the following form of the θ-field

$$\theta(x) = m\frac{2\pi}{L}x + \theta_0 \tag{2}$$

   Find the lowest energy in each sector (which is labeled by m and the angular momentum of θ_0).

   (b) The angular momentum (*ie* the canonical momentum) of θ_0 is the total number N of bosons. What is the total number of bosons for the lowest energy state in each sector. (Hint: how to find the canonical momentum of θ_0 from the coordinate space Lagrangian?)

   (c) Now we like to figure out what is the total momentum for the lowest energy state in each sector.

   One way to do so is to note that translating by Δx will shift the field $\theta(x)$ by $\Delta\theta = m\frac{2\pi}{L}\Delta x$. Shifting $\theta(x)$ by $\Delta\theta$ is a $U(1)$ transformation which will change the quantum state with N bosons by a phase $\mathrm{e}^{\mathrm{i}N\Delta\theta}$. Using the above information to find the total momentum for the lowest energy state in each sector.

   Another way to do so is to consider

$$\theta(x,t) = m\frac{2\pi}{L}(x - x_0(t)) + \theta_0 \tag{3}$$

   and assume only x_0 is dynamical (*ie* depend on time). Find the effective Lagrangian of x_0 and its canonical momentum. Then use such information to find the total momentum for the lowest energy state in each sector.

   m in $\theta(x)$ corresponds to the $U(1)$ symmetry twist. The dependence of total momentum on m indicates the ***pumping phenomenon****: Twisting the boundary condition by the $U(1)$ symmetry can generate momentum, the quantum number of the translation symmetry. This is called a mixed anomaly between translation and $U(1)$ symmetries. We see that the presence of the topological term implies the mixed anomaly. At classical level, the topological term does not affect the equation of motion and has no effect. At quantum level, the topological term has a big effect on the low energy dynamics of the non-linear σ-model: The non-linear σ-model must be gapless for any values of V_1 and V_2, or even after we add additional interaction terms, such as $(\partial_x\theta)^4$. See next problem.*

2. (25 pts) **Mixed anomaly and ground state degeneracy or gaplessness**

In the class, we discussed the mixed anomaly between translation symmetry and $U(1)$ symmetry. Physically, the mixed anomaly can be described by a **pumping phenomenon**: Twisting the boundary condition by the $U(1)$ symmetry can generate momentum, the quantum number of the translation symmetry.

To be concrete, let us consider a 1d bosonic system on a ring of L site, which is described by the following spin-1/2 Hamiltonian

$$H_0 = \sum_{i=0}^{L-1} H_{i,i+1}, \quad i \sim i+L \tag{4}$$

where the Hamiltonian is a sum of local terms

$$H_{i,i+1} = (-tb_i b_{i+1}^\dagger + h.c.) + \mu n_i + V n_i n_{i+1} \tag{5}$$

b_i (b_j) is the boson annihilation (creation) operator, and n_i and the boson density operator:

$$[b_i, b_j] = 0, \quad [b_i, b_j^\dagger] = \delta_{i,j}, \quad n_i = b_i^\dagger b_j, \tag{6}$$

and satisfy

$$[n_i, b_j] = -b_j \delta_{ij}, \quad [n_i, b_j^\dagger] = b_j^\dagger \delta_{ij}. \tag{7}$$

(a) Let $|\Phi\rangle$ be the ground state of H_0. Let $|\Phi^\theta\rangle = \mathrm{e}^{\mathrm{i}\theta \sum_{i=0}^{L-1} \frac{i}{L} n_i}|\Phi\rangle$. Let N_0 be the total number of bosons in the ground state $|\Phi\rangle$: $\sum_i n_i|\Phi\rangle = N_0|\Phi\rangle$. Let K_0 be the crystal momentum of $|\Phi\rangle$: $T|\Phi\rangle = \mathrm{e}^{\mathrm{i}K_0}|\Phi\rangle$ where T is the translation operator defined by $T b_i T^\dagger = b_{i+1}$. Show that $|\Phi^\theta\rangle$ is an eigenstate of the translation operator T when $\theta = 0$ mod 2π. Show that the eigenvalue is $\mathrm{e}^{\mathrm{i}(K_0+\Delta K)}$ and find ΔK.

($\mathrm{e}^{\mathrm{i}\theta \sum_{i=0}^{L-1} \frac{i}{L} n_i}$ *is a spacial dependent $U(1)$ transformation, which generates the $U(1)$ symmetry twist. A $\Delta K \neq 0$ mod 2π indicates a pumping phenomenon and a mixed anomaly. It also implies that $|\Phi^\theta\rangle$ (for $\theta = 0$ mod 2π) and $|\Phi\rangle$ are orthogonal many-body states. However, when $\Delta K = 0$ mod 2π, $|\Phi^\theta\rangle$ (for $\theta = 0$ mod 2π) and $|\Phi\rangle$ may be the same state.*)

(b) Next, we like to show that $|\Phi^\theta\rangle$ (for $\theta = 0$ mod 2π) and $|\Phi\rangle$ have almost the same energy. This allows us to conclude that a mixed anomaly implies gaplessness or ground state degeneracy.

To achieve this goal, first note that, for $\theta = 0$ mod 2π,

$$\langle\Phi^\theta|H_0|\Phi^\theta\rangle = L\langle\Phi^\theta|H_{i,i+1}|\Phi^\theta\rangle = L\langle\Phi|H_{i,i+1}^\theta|\Phi\rangle = L\epsilon_{i,i+1}^\theta, \tag{8}$$

where

$$H_{i,i+1}^\theta \equiv \mathrm{e}^{-\mathrm{i}\theta \sum_{i=0}^{L-1} \frac{i}{L} n_i} H_{i,i+1} \mathrm{e}^{\mathrm{i}\theta \sum_{i=0}^{L-1} \frac{i}{L} n_i}, \quad \epsilon_{i,i+1}^\theta, \equiv \langle\Phi|H_{i,i+1}^\theta|\Phi\rangle \tag{9}$$

Find $H_{i,i+1}^\theta$. Note that $H_{i,i+1}^\theta$ is well defined for any θ. We do not need to require $\theta = 0$ mod 2π.

(c) Show that $\frac{\mathrm{d}^2 H_{i,i+1}^\theta}{\mathrm{d}\theta^2} = O(L^{-2})$ and $\frac{\mathrm{d}^2 \epsilon_{i,i+1}^\theta}{\mathrm{d}\theta^2} = O(L^{-2})$.

(d) Show that $\epsilon_{i,i+1}^{\theta=2\pi} \geq \epsilon_{i,i+1}^{\theta=0}$ and $\epsilon_{i,i+1}^{\theta=-2\pi} \geq \epsilon_{i,i+1}^{\theta=0}$. Show that $\epsilon_{i,i+1}^{\theta=2\pi} - \epsilon_{i,i+1}^{\theta=0} = O(L^{-2})$. (Hint: $\epsilon_{i,i+1}^{\theta=0}$ is the ground state energy per site.)

(e) Show that, for $\theta = 0 \mod 2\pi$,

$$\langle \Phi^\theta | H_0 | \Phi^\theta \rangle - \langle \Phi | H_0 | \Phi \rangle = O(L^{-1}). \tag{10}$$

Show that when the ground state has $\frac{p}{q}$ bosons per site, the ground state is at least (nearly) q-fold degenerate. ("Nearly degenerate" means that the ground state energy splitting approaches zero as $L \to \infty$.) This implies that when the ground state has an irrational number of bosons per site, the ground state is gapless regardless the strength and the form of the interaction.

(*The non-linear σ-model field theory corresponds to this case. This result was first obtained in E. Lieb, T. Schultz, and D. Mattis, Annals of Physics 16, 407 (1961).*)

Term paper

The term paper is about four pages. It may be a short review article or a short article about a problem you worked on. The goal is to tell a story, such as an experimental phenomenon, a theoretical frame work, etc.

Please e-mail me your topic (with subject line: 8.513 term paper topic). The term paper is due on Dec 9 (you upload your term paper to canvas). You are encouraged to send me a draft around Nov 30 (with subject line: 8.513 term paper draft), so I can give some feedback.

Some possible term paper topics:

1. Magnetic orders and their spin waves.
2. Boltzmann transport equation for bands with Berry curvature.
3. Chern insulators (IQH states) and edge states.
4. Graphene and Dirac fermions (and how to gap them).
4. Bilayer Graphene and its band structure.
5. Weyl semi metals.
8. Fractional quantum Hall states.
9. Conformal field theories
11. Haldane phase (or AKLT state) of spin-1 chain.
12. Topological superconductors and Majorana zero modes.
13. Quantum critical point and critical exponents.
14. Or any other topic in condensed matter physics.

8.513 Modern Quantum Many-body Physics for Condensed Matter Systems
Fall 2021