Massachusetts Institute of Technology

Department of Physics

Course: 8.701 — Introduction to Nuclear and Particle Physics
Term: Fall 2020
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Problem Set 3

handed out October 7th, 2020

Electron-positron annihilation and pair production

Note: You can also find discussions on these problems from Modern Particle Physics, Mark Thomson 6.2 - 6.3.

Problem 1: Helicity combinations [20 points]

To calculate the $e^+e^- \to \mu^+\mu^-$ cross section, the matrix element needs to be evaluated taking all possible spin states into account. Draw all possible initial state helicity combinations. How many helicity combinations are there in total?

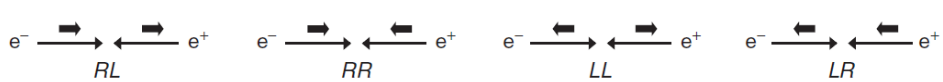


Fig. 6.3 The four possible helicity combinations in the e^+e^- initial state.

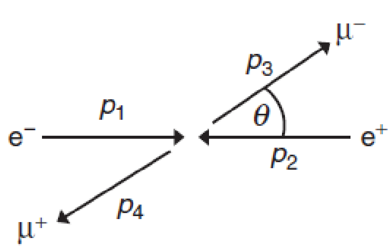
Problem 2:   Spinor [20 points]

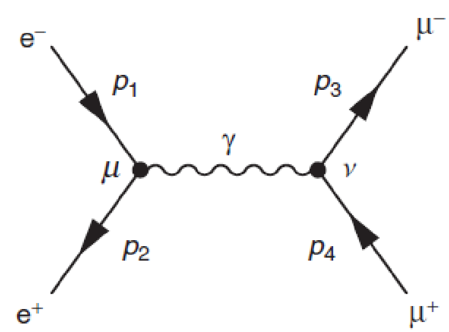
Show the four-momenta of the initial and final state particles in $e^+e^- \to \mu^+\mu^-$ in the limit where masses of particles can be neglected. Without loss of generality, take μ^+ and μ^- to be produced with azimuthal angles of $\phi = 0$ and $\phi = \pi$, respectively. Give all spinors for the initial and final state particles and lable the helicity configuration.

- Taking the ultra-relativistic limit in the center of mass frame, we can write the four momenta as:

$$\begin{aligned} p_1 &= (E, 0, 0, E) \\ p_2 &= (E, 00, -E) \\ p_3 &= (E, E\sin\theta, 0, E\cos\theta) \\ p_4 &= (E, -E\sin\theta, 0, -E\cos\theta) \end{aligned}$$

where the index for particles are shown in the following figure:





In the ultra-relativistic limit, the helicity eigenstates of particles and anti-particles are:

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix},\ u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix},\ v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix},\ v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix},$$

where $\uparrow$ denotes right-handed helicity, $c \equiv \cos\frac{\theta}{2}$, and $s \equiv \sin\frac{\theta}{2}$. Plug in $(\theta, \phi) = (0, 0)$ for electron, (π, π) for positron, $(\theta, 0)$ for muon and $(\pi - \theta, \pi)$ for anti-muon, we write out the spinors for the initial and final state particles:

$$u_\uparrow(p_1) = \sqrt{E}\begin{pmatrix}1\\0\\1\\0\end{pmatrix},\ u_\downarrow(p_1) = \sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix},\ v_\uparrow(p_2) = \sqrt{E}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix},\ v_\downarrow(p_2) = \sqrt{E}\begin{pmatrix}0\\-1\\0\\-1\end{pmatrix}.$$

$$u_\uparrow(p_3) = \sqrt{E}\begin{pmatrix}c\\s\\c\\s\end{pmatrix},\ u_\downarrow(p_3) = \sqrt{E}\begin{pmatrix}-s\\c\\s\\-c\end{pmatrix},\ v_\uparrow(p_4) = \sqrt{E}\begin{pmatrix}c\\s\\-c\\-s\end{pmatrix},\ v_\downarrow(p_4) = \sqrt{E}\begin{pmatrix}s\\-c\\s\\-c\end{pmatrix}.$$

Problem 3: Currents [20 points]

The matrix element for a particular helicity combination is $M = -\frac{e^2}{s} j_e \cdot j_\mu$ with s being the four times the beam energy and j_e and j_μ the four vector currents. The muon current, $j_\mu^\nu = \bar{u}\gamma^\nu v$, needs to be evaluate for the possible helicity combinations. Find the four components of the muon current for all helicity combinations. Which combinations yield non-zero four-vector currents? Repeat the discussion for the electron currents.

- The muon and electron currents take the form:

$$j_\mu^\nu = \bar{u}(p_3)\gamma^\nu v(p_4) \quad (1)$$
$$j_e^\mu = \bar{v}(p_2)\gamma^\mu u(p_1) \quad (2)$$

For any two spinors ψ and ϕ, we have the following identities:

$$\overline{\psi}\gamma^0\phi = \psi^\dagger\gamma^0\gamma^0\phi = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4,$$
$$\overline{\psi}\gamma^1\phi = \psi^\dagger\gamma^0\gamma^1\phi = \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1,$$
$$\overline{\psi}\gamma^2\phi = \psi^\dagger\gamma^0\gamma^2\phi = -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1),$$
$$\overline{\psi}\gamma^3\phi = \psi^\dagger\gamma^0\gamma^3\phi = \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2.$$

Using the specific form of spinors found in problem 2, we get:

$$j_{\mu,RL} = \overline{u}_{\uparrow}(p_3)\gamma^{\nu}v_{\downarrow}(p_4) = 2E(0, -\cos\theta, i, \sin\theta),$$
$$j_{\mu,RR} = \overline{u}_{\uparrow}(p_3)\gamma^{\nu}v_{\uparrow}(p_4) = (0, 0, 0, 0),$$
$$j_{\mu,LL} = \overline{u}_{\downarrow}(p_3)\gamma^{\nu}v_{\downarrow}(p_4) = (0, 0, 0, 0),$$
$$j_{\mu,LR} = \overline{u}_{\downarrow}(p_3)\gamma^{\nu}v_{\uparrow}(p_4) = 2E(0, -\cos\theta, -i, \sin\theta).$$

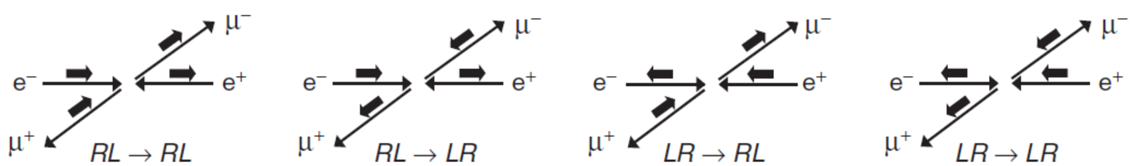
$$j_{e,RL} = \overline{v}_{\downarrow}(p_2)\gamma^{\mu}u_{\uparrow}(p_1) = 2E(0, -1, -i, 0),$$
$$j_{e,LR} = \overline{v}_{\uparrow}(p_2)\gamma^{\mu}u_{\downarrow}(p_1) = 2E(0, -1, i, 0).$$

also, $j_{e,RR} = j_{e,LL} = 0$.

Problem 4: Spin [20 points]

Define the z-axis to be the direction of the incoming electron beam. What are the spins of the combined spins of the e^+ and e^- for the non-zero matrix elements? Express the spin states of the $\mu^+\mu^-$ system in terms of the eigenstates of spin-operator $\hat{S}_z = \frac{1}{2}\Sigma_z$.

- There are four helicity configurations with non-zero matrix elements:



The four helicity combinations for $e^+e^- \rightarrow \mu^+\mu^-$ that in the limit $E \gg m$ give non-zero matrix elements.

Note that in either case, the electron spins are aligned, so are the muon spins. Defining the z-axis to be the direction of the incoming electron beam, we deduce that the combined spin for e^- and e^+ should be either 1 or -1, which means that they are substates of a spin-1 system. To express the spin states for the $\mu^-\mu^+$ system in the eigenbasis of $\hat{\mathbf{S}}$, the spin operator for the e^-e^+ system, we first calculate the spin operator for $\mu^-\mu^+$:

$\hat{S}_n = \mathbf{n} \cdot \hat{\mathbf{S}} = \sin \hat{\mathbf{S}}_{\mathbf{x}} + \cos \hat{\mathbf{S}}_{\mathbf{z}} = \frac{1}{2}\sin\theta(\hat{\mathbf{S}}_{+} + \hat{\mathbf{S}}_{-}) + \cos\theta \hat{\mathbf{S}}_{\mathbf{z}}$

The $\mu^-\mu^+$ spin states, which are eigenstates of $\hat{S}_n$, satisfy:

$$\hat{S}_n \left|1, 1\right\rangle_\theta = \left|1, 1\right\rangle_\theta \tag{3}$$
$$\hat{S}_n \left|1, -1\right\rangle_\theta = -\left|1, -1\right\rangle_\theta . \tag{4}$$

We write them in the eigenbasis of $\hat{S}$:

$$|1,1\rangle_\theta = \alpha\,|1,1\rangle + \beta\,|1,0\rangle + \gamma\,|1,-1\rangle$$
$$|1,-1\rangle_\theta = \tilde{\alpha}\,|1,1\rangle + \tilde{\beta}\,|1,0\rangle + \tilde{\gamma}\,|1,-1\rangle\,,$$

and plug them back into Equation 3. Two relations that come in handy for this calculation are:

$$\hat{S}_+\,|1,m\neq 1\rangle = \sqrt{2}\,|1,m+1\rangle$$
$$\hat{S}_-\,|1,m\neq -1\rangle = \sqrt{2}\,|1,m-1\rangle$$

Do the algebra and we find:

$$|1,1\rangle_\theta = \frac{1}{2}(1-\cos\theta)\,|1,1\rangle + \frac{1}{\sqrt{2}}\sin\theta\,|1,0\rangle + \frac{1}{2}(1+\cos\theta)\,|1,-1\rangle$$
$$|1,-1\rangle_\theta = \frac{1}{2}(1+\cos\theta)\,|1,1\rangle - \frac{1}{\sqrt{2}}\sin\theta\,|1,0\rangle + \frac{1}{2}(1-\cos\theta)\,|1,-1\rangle\,.$$

A quick sanity check: ${}_\theta\langle 1,1|1,-1\rangle_\theta = \frac{1}{2}\sin^2\theta - \frac{1}{2}\sin^2\theta = 0$.

Problem 5: Photon Decay [20 points]

Consider the following process $e^+e^- \to \gamma\gamma$. Discuss the kinematics of this process with four-momenta. What can you say about the spin configuration and the photon current?

- In the center-of-mass frame where the z-axis is along the position beam, we can write the four momenta of the particles in the following form:

$$\begin{aligned} p_{e^+} &= (\sqrt{p^2+m_e^2}, 0, 0, p) \\ p_{e^-} &= (\sqrt{p^2+m_e^2}, 0, 0, -p) \\ p_{\gamma_1} &= \sqrt{p^2+m_e^2}(1, \sin\theta, 0, \cos\theta) \\ p_{\gamma_2} &= \sqrt{p^2+m_e^2}(1, -\sin\theta, 0, -\cos\theta) \end{aligned}$$

We know that photons are massless spin-1 particles, which means they have only 2 polarizations: $|1,-1\rangle$ and $|1,1\rangle$. If the spins of the two photons align, then they add up to states of $|j,\pm 2\rangle$, which requires $j \geq 2$. This violates spin conservation from adding up 2 spin-$\frac{1}{2}$ particles in the initial state. Therefore, the photons must have anti-parallel spin, which corresponds to the LL or RR helicity configuration. In this case, the electron-positron system should also have anti-parallel spins, which make up either the $|0,0\rangle$ or the $|1,0\rangle$ state. An addition constraint from CP-conservation limits the electron-positron system to be in the $|0, 0\rangle$ state to match the CP-even di-photon final state.

Based on the discussion on photon spin states, it is clear that 2 photons could not add up to $|j, \pm 1\rangle$ states. Therefore, it is impossible to have a one-photon-to-two-photons vertex at tree-level.