8.701

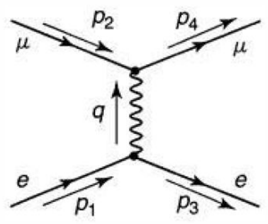
Introduction to Nuclear and Particle Physics

Markus Klute - MIT

4. QED

4.7 Casimir's Trick

Polarized particles



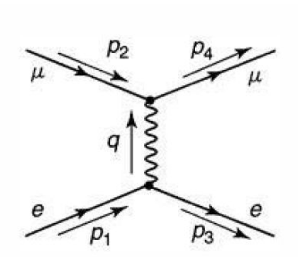
A typical QED amplitude looks like

$$\mathcal{M} \sim [\bar{u}_1 \Gamma^\mu v_2] \epsilon_{3\mu}$$

To get a number for M we will need to use explicit forms for the wavefunctions of the external particles

If all external particles have a known polarization, this might be reasonable. Often, we are interested in unpolarized particles.

Spin-Averaged Amplitudes



If we do not care about the polarization of the particles, we need to

Average over the polarizations of the initial-state particles

Sum over polarizations of the final-state particles in the squared amplitude

We call this the spin-averaged amplitude

$$\left\langle |\mathcal{M}|^2 \right\rangle$$

Spin Sums

Suppose that we have $\mathcal{M} \sim [\bar{u}_1 \Gamma u_2]$

Then

$$
\begin{aligned}
|\mathcal{M}|^2 &\sim [\bar{u}_1 \Gamma u_2] [\bar{u}_1 \Gamma u_2]^* \\
&\sim [\bar{u}_1 \Gamma u_2] \left[u_1^\dagger \gamma^0 \Gamma u_2\right]^\dagger \\
&\sim [\bar{u}_1 \Gamma u_2] \left[u_2^\dagger \Gamma^\dagger \gamma^{0\dagger} u_1\right] \\
&\sim [\bar{u}_1 \Gamma u_2] \left[u_2^\dagger \gamma^0 \gamma^0 \Gamma^\dagger \gamma^0 u_1\right] \\
&\sim [\bar{u}_1 \Gamma u_2] [\bar{u}_2 \bar{\Gamma} u_1]
\end{aligned}
$$

$$
|\mathcal{M}|^2 \sim [\bar{u}_1 \Gamma u_2] [\bar{u}_2 \bar{\Gamma} u_1]
$$

Spin Sums

Applying the completeness relation

$$\sum_{s_i=1,2} u_i^{s_i}\bar{u}_i^{s_i} = (\not{p}_i + m_i)$$

and summing over the spins of particle 2

$$\begin{aligned}\sum_{s_2} |\mathcal{M}|^2 \quad &\sim \quad [\bar{u}_1\Gamma(\not{p}_2 + m_2)\bar{\Gamma}u_1] \\ &\sim \quad [\bar{u}_1 Q u_1]\end{aligned}$$

Spin Sums

We can represent the matrix multiplication with summations over indices

$$\begin{aligned} [\bar{u}_1 Q u_1] &= (\bar{u}_1)_i \, Q_{ij} \, (u_1)_j \\ &= Q_{ij} \, (u_1 \bar{u}_1)_{ji} \\ &= [Q \, (u_1 \bar{u}_1)]_{ii} \\ &= \mathrm{Tr} \, [Q(u_1 \bar{u}_1)] \end{aligned}$$

Finally, we apply the completeness relation again

$$\sum_{s_1} |\mathcal{M}|^2 \sim \mathrm{Tr} \, [Q(\not{p}_1 + m_1)]$$

Summary

We have

$$
\begin{aligned}
\mathcal{M} &\sim [\bar{u}_1 \Gamma u_2] \\
\Rightarrow \left\langle |\mathcal{M}|^2 \right\rangle &\sim \frac{1}{2} \operatorname{Tr}\left[\Gamma(\not{p}_2 + m_2)\bar{\Gamma}(\not{p}_1 + m_1)\right]
\end{aligned}
$$

The ½ comes from averaging over initial spins, assuming exactly one of u1 and u2 corresponds to an initial-state particle. If they are both initial states (pair annihilation) the factor is ¼ and if neither is initial-state (pair production, the factor is 1.

Casimir's Trick

The trick to calculate spin-averaged amplitudes in terms of traces is known as Casimir's Trick

$$\sum_{\text{all spins}} [\bar{u}_a \Gamma_1 u_b] [\bar{u}_a \Gamma_2 u_b]^* = \text{Tr}\left[\Gamma_1 (\not{p}_b + m_b) \bar{\Gamma}_2 (\not{p}_a + m_a)\right]$$

If antiparticle spinors v are present in the spin sum, we use

$$\sum_{s_i=1,2} v_i^{s_i} \bar{v}_i^{s_i} = (\not{p}_i - m_i)$$

Traces

$$
\begin{aligned}
\mathrm{Tr}(A+B) &= \mathrm{Tr}(A)+\mathrm{Tr}(B) \\
\mathrm{Tr}(\alpha A) &= \alpha \mathrm{Tr}(A) \\
\mathrm{Tr}(AB) &= \mathrm{Tr}(BA) \\
\mathrm{Tr}(ABC) &= \mathrm{Tr}(CAB) = \mathrm{Tr}(BCA)
\end{aligned}
$$

Gamma Matrices

$$
\begin{aligned}
g_{\mu\nu} g^{\mu\nu} &= 4 \\
\{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu}(\times \mathbf{I})
\end{aligned}
$$

$$
\begin{aligned}
\gamma_\mu \gamma^\nu \gamma^\mu &= \gamma_\mu \left(2g^{\mu\nu} - \gamma^\mu \gamma^\nu\right) \\
&= 2\gamma^\nu - \gamma_\mu \gamma^\mu \gamma^\nu \\
&= 2\gamma^\nu - 4\gamma^\nu \\
&= -2\gamma^\nu
\end{aligned}
$$

Trace identities

$$\begin{aligned}\mathrm{Tr}(\gamma^\mu\gamma^\nu) &= \mathrm{Tr}\left(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right)/2 \\ &= \mathrm{Tr}(2g^{\mu\nu})/2 \\ &= g^{\mu\nu}\mathrm{Tr}(1) \\ &= 4g^{\mu\nu}\end{aligned}$$

$$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4\left(g^{\mu\nu}g^{\lambda\sigma} - g^{\mu\lambda}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\lambda}\right)$$

Trace with γ^5

$$\mathrm{Tr}(\gamma^5) = 0$$

Since $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

$$\mathrm{Tr}(\gamma^5\gamma^\mu) = 0$$

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda) = 0$$

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu) = 0$$

Trace with γ^5

Only with 4 (or more) other γ-matrices can we find nonzero traces involving γ^5

$$\mathrm{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma) = 4i\epsilon^{\mu\nu\lambda\sigma}$$

with the totally antisymmetric tensor

$$\epsilon^{\mu\nu\lambda\sigma} \equiv \begin{cases} -1 & \text{for } \textit{even} \text{ permutations of } 0123 \\ +1 & \text{for } \textit{odd} \text{ permutations of } 0123 \\ 0 & \text{if any 2 indices are the same} \end{cases}$$