Massachusetts Institute of Technology

Department of Physics

Course:	8.701 – Introduction to Nuclear and Particle Physics
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Discussion Problems

from recitation on **October 1st, 2020**

Problem 1: Field strength tensor

Derive

$$\nabla \cdot \mathbf{E} = 4\pi\rho \tag{1}$$

and

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J} \tag{2}$$

from

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}\mathrm{J}^\nu \tag{3}$$

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(a) Start with Eq. 7.73: $\partial_\mu F^{\mu\nu} = (4\pi/c)J^\nu$. For the case $\nu = 0$:

$$\begin{aligned}
\partial_\mu F^{\mu 0} &= \frac{\partial}{\partial x^0}F^{00} + \frac{\partial}{\partial x^1}F^{10} + \frac{\partial}{\partial x^2}F^{20} + \frac{\partial}{\partial x^3}F^{30} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \nabla \cdot \mathbf{E} \\
&= \frac{4\pi}{c}J^0 = \frac{4\pi}{c}c\rho = 4\pi\rho. \quad \text{So} \quad \nabla \cdot \mathbf{E} = 4\pi\rho. \checkmark
\end{aligned}$$

For the case $\nu = 1$:

$$\begin{aligned}
\partial_\mu F^{\mu 1} &= \frac{\partial}{\partial x^0}F^{01} + \frac{\partial}{\partial x^1}F^{11} + \frac{\partial}{\partial x^2}F^{21} + \frac{\partial}{\partial x^3}F^{31} = \frac{\partial(-E_x)}{\partial(ct)} + \frac{\partial B_z}{\partial y} + \frac{\partial(-B_y)}{\partial z} \\
&= -\frac{1}{c}\frac{\partial E_x}{\partial t} + \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) = \left[-\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + (\nabla \times \mathbf{B})\right]_x \\
&= \frac{4\pi}{c}J^1 = \frac{4\pi}{c}J_x.
\end{aligned}$$

This is the x component of

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{J} \quad \checkmark$$

(the y component comes from $\nu = 2$, and the z component from $\nu = 3$).

Problem 2: Continuity equation

Show that the continuity equation

$$\partial_\mu \mathrm{J}^\mu = 0, \tag{4}$$

which follows from the antisymmetry of $F^{\mu\nu}$, enforces conservation of charge.

$$0 = \partial_\mu J^\mu = \partial_0 J^0 + \partial_1 J^1 + \partial_2 J^2 + \partial_3 J^3 = \frac{\partial(c\rho)}{\partial(ct)} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}$$
$$\implies \quad \frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}.$$

Integrate over some volume V, with surface S:

$$\int_V \frac{\partial \rho}{\partial t} d\tau = \frac{d}{dt}\int_V \rho \, d\tau = \frac{dQ}{dt} = -\int_V \nabla \cdot \mathbf{J} \, d\tau = -\int_S \mathbf{J} \cdot d\mathbf{A}$$

(where Q is the total charge in V, and I used the divergence theorem in the last step). This says that the rate of change of the charge in V is minus the flux of charge out through the surface—no charge simply disappears or is created from nothing. In particular, if we pick a volume such that $\mathbf{J}$ is *zero* at the surface, then Q is constant.