8.701

Introduction to Nuclear and Particle Physics

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4. QED

4.1 Free Wave Equations

Klein-Gordon wave equation

Relativistic energy-momentum relation

$$E^2 = p^2c^2 + m^2c^4$$

Use quantum mechanical operators

$$E_{\text{op}} = i\hbar\frac{\partial}{\partial t}, \quad p_{\text{op}} = -i\hbar\nabla = -i\hbar\frac{\partial}{\partial \mathbf{r}}$$

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \nabla^2\psi - \frac{m^2c^2}{\hbar^2}\psi$$

$$\boxed{\frac{\partial^2\psi}{\partial t^2} = (\nabla^2 - m^2)\psi}$$

Weyl equation

Attempt to find 1st order equation in both derivatives

$$\frac{\partial \psi}{\partial t} = \pm \left(\sigma_1 \frac{\partial \psi}{\partial x} + \sigma_2 \frac{\partial \psi}{\partial y} + \sigma_3 \frac{\partial \psi}{\partial z} \right) = \pm \boldsymbol{\sigma} \cdot \frac{\partial}{\partial \mathbf{r}} \psi$$

with σ’s being unknown constants. To satisfy the Klein-Gordon eq., we square and equate the coefficients

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = 1 \qquad \sigma_1 \sigma_2 + \sigma_2 \sigma_1 = 0, \quad \text{etc.} \qquad m = 0$$

Pauli matrices

The σ's can not be numbers as they do not commute, but can be represented by matrices.

The equations define 2x2 Pauli matrices.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Two Weyl equations with spinors are separate solutions

$$E\chi = -\boldsymbol{\sigma} \cdot \mathbf{p}\chi$$

$$E\phi = +\boldsymbol{\sigma} \cdot \mathbf{p}\phi$$

Dirac equations

Including mass terms

$$E\psi = (\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m)\psi$$

with a 4 component spinor (particle, antiparticle, and two spin states each) and with **α** and **β** being 4x4 matrices

$$\alpha = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dirac equations

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$$E\psi = (\boldsymbol{\alpha} \cdot \mathbf{p} + \beta m)\psi$$

with a 4 component spinor (particle, antiparticle, and two spin states each) and with **α** and **β** being 4x4 matrices

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dirac equations

$$\boxed{\left(i\gamma_\mu \frac{\partial}{\partial x_\mu} - m\right)\psi = 0}$$

$$\gamma_k = \beta\alpha_k = \begin{pmatrix} 0 & \boldsymbol{\sigma}_k \\ -\boldsymbol{\sigma}_k & 0 \end{pmatrix}, \quad k = 1, 2, 3 \quad \text{and} \quad \gamma_4 = \beta$$