# Massachusetts Institute of Technology

## Department of Physics

Course: 8.701 — Introduction to Nuclear and Particle Physics
Term: Fall 2020
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# Problem Set 2

handed out September 23rd, 2020

Note: Throughout the solutions we set $\hbar = c = 1$.

## Problem 1: Complex scalar field [20 points]

The Lagrangian for a complex scalar field $\mathcal{L}_s = \frac{1}{2}(\partial_\mu \phi)^*(\partial^\mu \phi) - \frac{1}{2}m^2\phi^*\phi$ possesses a global $U(1)$ symmetry. Use Noether's theorem to identify the conserved current.

Under $U(1)$ symmetry, the field Lagrangian is invariant under the transformation:

$$\phi \rightarrow e^{-i\alpha}\phi$$
$$\phi^* \rightarrow e^{i\alpha}\phi^*,$$

where

$$\frac{\delta\phi}{\delta\alpha} = \frac{\delta(e^{-i\alpha}\phi)}{\delta\alpha} = \frac{\delta(1 - i\alpha - \alpha^2 + ...)}{\delta\alpha}\phi = -i\phi$$
$$\frac{\delta\phi^*}{\delta\alpha} = \frac{\delta(e^{i\alpha}\phi^*)}{\delta\alpha} = \frac{\delta(1 + i\alpha - \alpha^2 + ...)}{\delta\alpha}\phi^* = i\phi^*.$$

So, the conserved current is:

$$\begin{aligned} J^\mu &= \tfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\tfrac{\delta\phi}{\delta\alpha} + \tfrac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)^*}\tfrac{\delta\phi^*}{\delta\alpha} \\ &= \tfrac{1}{2}(\partial^\mu\phi^*)(-i\phi) + \tfrac{1}{2}(\partial^\mu\phi)i\phi^* \\ &= \tfrac{-i}{2}\left(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi\right). \end{aligned}$$

## Problem 2: $A \rightarrow B + C$ [20 points]

Show that $|\vec{p}_{CM}| = \frac{1}{2m_A}\sqrt{(m_A^2 - (m_B + m_C)^2)(m_A^2 - (m_B - m_C)^2)}$ for $A \rightarrow B + C$.

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In the center of mass frame, we have:

$$p_A = (m_A, \vec{0})$$
$$p_B = (\sqrt{|\mathbf{p}|^2 + m_B^2}, \mathbf{p})$$
$$p_C = (\sqrt{|\mathbf{p}|^2 + m_C^2}, -\mathbf{p}).$$

So,

$$m_A^2 = (\sqrt{|\mathbf{p}|^2 + m_B^2} + \sqrt{|\mathbf{p}|^2 + m_C^2})^2,$$

expanding the binomial on the right side and rearrange the terms:

$$\frac{m_A^2 - m_B^2 - m_C^2}{2} - |\mathbf{p}|^2 = \sqrt{(|\mathbf{p}|^2 + m_B^2)(|\mathbf{p}|^2 + m_C^2)},$$

square both sides again and expand & cancel terms:

$$\left(\frac{m_A^2 - m_B^2 - m_C^2}{2}\right)^2 - (m_A^2 - m_B^2 - m_C^2)|\mathbf{p}|^2 = (m_B^2 + m_C^2)|\mathbf{p}|^2 + m_B^2 m_C^2$$
$$|\mathbf{p}|^2 = \frac{1}{4m_A^2}\left[(m_A^2 - m_B^2 - m_C^2)^2 - 4m_B^2 m_C^2\right]$$
$$|\mathbf{p}| = \frac{1}{2m_A}\sqrt{(m_A^2 - (m_B + m_C)^2)(m_A^2 - (m_B - m_C)^2)}$$

## Problem 3: $A + A \to A + A$ [20 points]

Draw all possible lowest-order diagrams for $A + A \to A + A$ in our toy theory and find the amplitude for this process assuming $m_B = m_C = 0$. Leave your answer in the form of an integral over the remaining four-momentum $q$.

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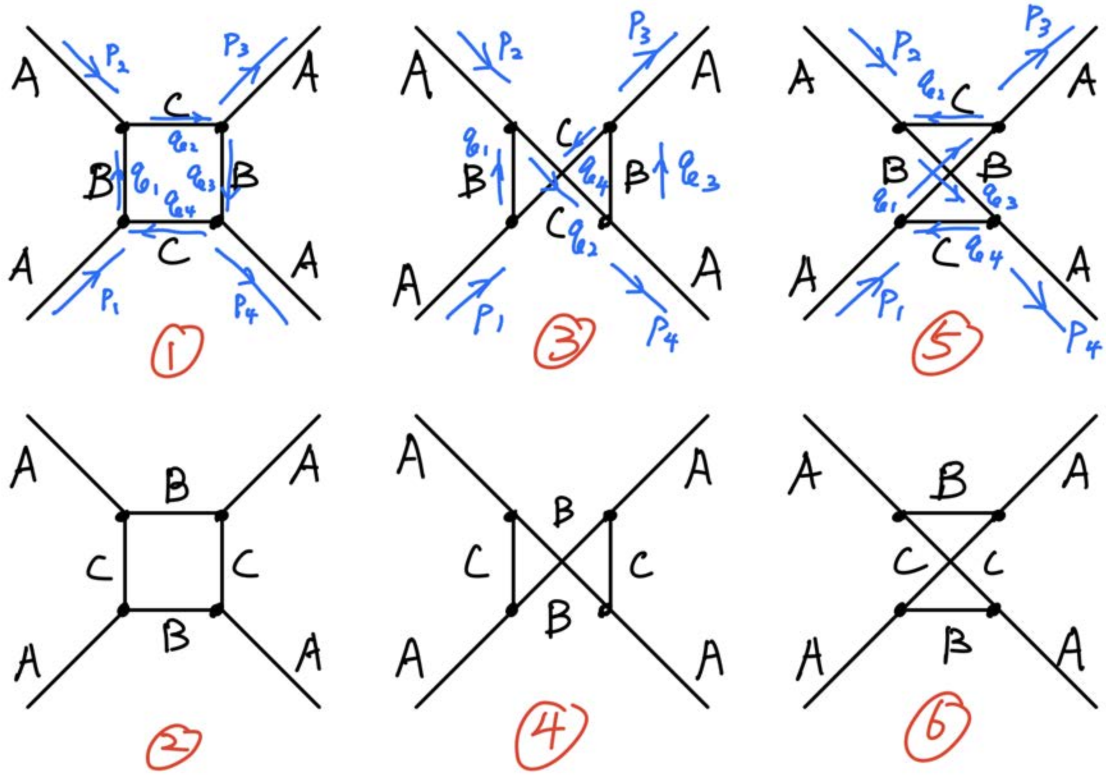


Figure 1: Lowest-order Feynman diagrams for $A + A \to A + A$.

There are 6 possible lowest-order diagrams, as shown in Figure 1.

For the transition amplitude, we first note that the 6 diagrams can be grouped into 3 pairs: 1-2, 3-4, 5-6. Within each pair, the diagrams are identical up to an exchange between B and C. Since $m_B = m_C = 0$ and we are ignoring spins in our toy model, the 2 diagrams in each pair should have the same amplitude.

Let us take a look at the first pair:

$$\begin{aligned} -i\mathcal{M}_1 = &\int (-ig)^4 \tfrac{i}{q_1^2}\tfrac{i}{q_2^2}\tfrac{i}{q_3^2}\tfrac{i}{q_4^2}(2\pi)^4\delta^4(p_1+q_4-q_1)(2\pi)^4\delta^4(p_2+q_1-q_2)\times \\ &(2\pi)^4\delta^4(q_2-q_3-p_3)(2\pi)^4\delta^4(q_3-p_4-q_4)\tfrac{d^4q_1}{(2\pi)^4}\tfrac{d^4q_2}{(2\pi)^4}\tfrac{d^4q_3}{(2\pi)^4}\tfrac{d^4q_4}{(2\pi)^4} \\ = &\; g^4\int \tfrac{\delta^4(p_1+q_4-q_1)\delta^4(p_2+q_1-q_2)\delta^4(q_2-q_3-p_3)\delta^4(q_3-p_4-q_4)}{q_1^2q_2^2q_3^2q_4^2} d^4q_1d^4q_2d^4q_3d^4q_4. \end{aligned}$$

Integrating out $q_4$ and $q_3$ brings $q_4 \Rightarrow q_1 - p_1$ and $q_3 \Rightarrow q_2 - p_3$:

$$-i\mathcal{M}_1 = g^4 \int \frac{\delta^4(p_2+q_1-q_2)\delta^4(q_2-p_3-p_4-q_1+p_1)}{q_1^2q_2^2(q_2-p_3)^2(q_1-p_1)^2} d^4q_1d^4q_2,$$

then integrate out $q_2$ ($q_2 \Rightarrow p_2 + q_1$) and let $q \equiv q_1$:

$$-i\mathcal{M}_1 = g^4 \int \frac{\delta^4(p_2 + q - p_3 - p_4 - q + p_1)}{q^2(p_2+q)^2(p_2+q-p_3)^2(q-p_1)^2} d^4q.$$

Cancel the last delta function and factor in $\frac{1}{(2\pi)^4}$, we get:

$$\mathcal{M}_1 = i\left(\frac{g}{2\pi}\right)^4 \int \frac{1}{q^2(q+p_2)^2(q+p_2-p_3)^2(q-p_1)^2} d^4q = \mathcal{M}_2.$$

Now, pair 3-4 is identical with pair 1-2 under $p_3 \leftrightarrow p_4$, and so is pair 5-6 under $p_2 \leftrightarrow -p_3$. Therefore:

$$\mathcal{M}_3 = i\left(\frac{g}{2\pi}\right)^4 \int \frac{1}{q^2(q+p_2)^2(q+p_2-p_4)^2(q-p_1)^2} d^4q = \mathcal{M}_4$$
$$\mathcal{M}_5 = i\left(\frac{g}{2\pi}\right)^4 \int \frac{1}{q^2(q-p_3)^2(q-p_3+p_2)^2(q-p_1)^2} d^4q = \mathcal{M}_6.$$

Summing them up, we acquire the full amplitude:

$$\mathcal{M} = 2i\left(\frac{g}{2\pi}\right)^4 \int \frac{1}{q^2(q-p_1)^2}\left\{\frac{1}{(q+p_2)^2(q+p_2-p_3)^2} + \frac{1}{(q+p_2)^2(q+p_2-p_4)^2} + \frac{1}{(q-p_3)^2(q-p_3+p_2)^2}\right. d^4q$$

## Problem 4: $A + A \to B + B$ [20 points]

Calculate $\frac{d\sigma}{d\Omega}$ for $A+A \to B+B$ a) in the center-of-mass frame and b) in the lab frame at lowest order. Assume $m_B = m_C = 0$ in a toy theory without spin. For a), calculate the total cross section $\sigma$. For b), determine the non-relativistic and ultra-relativiistic limits.

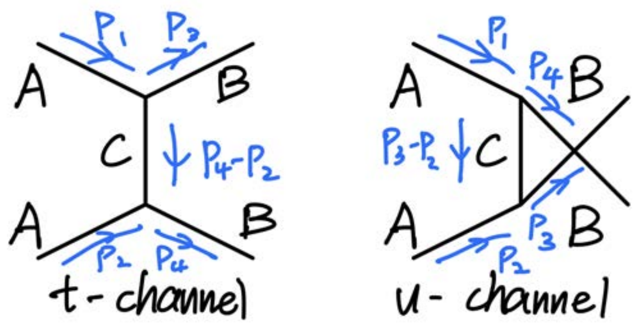


Figure 2: Lowest-order Feynman diagrams for $A + A \to B + B$.

There are 2 possible lowest-order diagrams for the $A + A \to B + B$scattering, as shown in Figure 2. We write out the external momenta in the center of mass frame:

$$p_1 = (E, \mathbf{p}_i);\ p_2 = (E, -\mathbf{p}_i)$$
$$p_3 = (E, \mathbf{p}_f);\ p_4 = (E, -\mathbf{p}_f).$$

With $m_C = 0$, the total amplitude is:

$$\begin{aligned}\mathcal{M} &= g^2\left\{\tfrac{1}{(p_4-p_2)^2} + \tfrac{1}{(p_3-p_2)^2}\right. \\ &= g^2\left\{\tfrac{1}{m_A^2-2E^2+2\mathbf{p}_i\cdot\mathbf{p}_f} + \tfrac{1}{m_A^2-2E^2-2\mathbf{p}_i\cdot\mathbf{p}_f}\right. \\ &= g^2\tfrac{2(m_A^2-2E^2)}{(m_A^2-2E^2)^2-4(\mathbf{p}_i\cdot\mathbf{p}_f)^2}.\end{aligned}$$

Well, $(\mathbf{p}_i \cdot \mathbf{p}_f)^2 = |\mathbf{p}_i|^2|\mathbf{p}_f|^2\cos^2\theta = (E^2 - m_A^2)(E^2)\cos^2\theta$. So,

$$\mathcal{M} = 2g^2\frac{(m_A^2 - 2E^2)}{(m_A^2 - 2E^2)^2 - 4E^2(E^2 - m_A^2)\cos^2\theta}.$$

The formula for 2-body scattering in the center-of-mass frame reads:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{S|\mathcal{M}|^2}{4E^2}\frac{|\mathbf{p}_f|}{|\mathbf{p}_i|}.$$

With $S = 1/2$, the differential cross section reduces to:

$$\frac{d\sigma}{d\Omega}_{CM} = \tfrac{g^4}{64\pi^2}\tfrac{1}{2E^2}\tfrac{E}{\sqrt{E^2-m_A^2}}\left[\tfrac{(m_A^2-2E^2)}{(m_A^2-2E^2)^2-4E^2(E^2-m_A^2)\cos^2\theta}\right]^2$$

$$\boxed{\frac{d\sigma}{d\Omega}_{CM} = \frac{g^4}{64\pi^2}\frac{1}{2E\sqrt{E^2 - m_A^2}}\left[\frac{(m_A^2 - 2E^2)}{(m_A^2 - 2E^2)^2 - 4E^2(E^2 - m_A^2)\cos^2\theta}\right]^2}$$

For the total cross section,

$$\sigma = \int \frac{d\sigma}{d\Omega}\,d\Omega = \int \frac{d\sigma}{d\Omega}\sin\theta\,d\theta\,d\phi = 2\pi A\int_0^\pi \frac{\sin\theta\,d\theta}{(a - b\cos^2\theta)^2} = 2\pi AI,$$

where:

$$\begin{aligned} A = & \quad \tfrac{g^4}{64\pi^2}\tfrac{1}{2E\sqrt{E^2-m_A^2}}(m_A^2-2E^2)^2 \\ a = & \quad (m_A^2-2E^2)^2 \\ b = & \quad 4E^2(E^2-m_A^2), \end{aligned}$$

and the integral evaluates to:

$$I = \int_{-1}^{1} \frac{\mathrm{d}x}{(a-bx^2)^2}\,(x \equiv \cos\theta) = \frac{1}{a}\left[\frac{1}{a-b} + \frac{1}{\sqrt{ab}}\tanh^{-1}\sqrt{\frac{b}{a}}\right].$$

Putting everything together, we get:

$$\boxed{\sigma_{CM} = \frac{g^4}{128\pi}\frac{1}{E^2(E^2-m_A^2)}\left[\frac{2E\sqrt{E^2-m_A^2}}{m_A^4} + \frac{1}{2E^2-m_A^2}\tanh^{-1}\frac{2E\sqrt{E^2-m_A^2}}{2E^2-m_A^2}\right]}$$

Now for the cross section in the lab frame where the target is at rest, the momenta are $(m \equiv m_A)$:

$$p_1 = (E, \mathbf{p}); \quad p_2 = (m, 0); \quad p_3 = E_3(1, \hat{p_3}); \quad p4 = E_4(1, \hat{p_4})$$

$|\mathcal{M}|^2$ should remain invariant, and:

$$\begin{aligned} (p_3-p_2)^2 = & \quad m^2 - 2mE_3 && (1) \\ (p_4-p_2)^2 = & \quad m^2 - 2mE_4 && (2) \\ = & \quad (p_1-p_3)^2 \\ = & \quad m^2 - 2(EE_3 - E_3|\mathbf{p}|\cos\theta), \end{aligned}$$

which gives:

$$E_4 = \frac{E_3}{m}(E - |\mathbf{p}|\cos\theta). \tag{3}$$

Citing the results from Problem 6.9 from Griffiths, the differential cross section for $A + A \to B + B$ should be, in general:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}_{\text{lab}} = \frac{1}{128\pi^2}\frac{|\mathcal{M}|^2|\mathbf{p}_3|}{m|\mathbf{p}|(E+m-|\mathbf{p}|\cos\theta)} \tag{4}$$

In the non-relativistic limit, $E \approx m \gg |\mathbf{p}|$, so Equation 1 and 2 reduce to $(p_3-p_2)^2 = (p_4-p_2)^2 = -m^2$, while Equation 3 reduces to $E_4 \approx E_3$. Also, since $E+m = E_3+E_4$, we conclude $E = E_3 = E_4 = m$. Thus, the amplitude reduces to:

$$\mathcal{M} = g^2\{\frac{1}{(p_4-p_2)^2} + \frac{1}{(p_3-p_2)^2} \quad = -\frac{2g^2}{m^2}.$$

Plug this back into Equation 4 and note that $|\mathbf{p}_3| = E_3 \approx m$, we get:

$$\frac{d\sigma}{d\Omega} = \frac{1}{128\pi^2} 4 \frac{g^4}{m^4} \frac{m}{m|\mathbf{p}|(2m)} = \frac{1}{64\pi^2} \frac{g^4}{m^5|\mathbf{p}|}.$$

In the non-relativistic limit, $|\mathbf{p}| \approx mv$, so:

$$\boxed{\lim_{v\to 0} \left.\frac{d\sigma}{d\Omega}\right|_{\text{lab}} = \left(\frac{g^2}{8\pi m^3}\right)^2 \frac{1}{v}}$$

In the ultra-relativistic limit, we have $E \approx |\mathbf{p}| \gg m$, so Equation 3 becomes:

$$E_4 = \frac{E_3}{m} E(1 - \cos\theta)$$

Also, in the ultra-relativistic limit, $E \approx E_3 + E_4$, therefore:

$$E_3 \left(1 + \frac{E(1-\cos\theta)}{m}\right) = E$$

$$E_3 = \frac{mE}{m + E(1-\cos\theta)}. \tag{5}$$

And,

$$E_4 = \frac{E_3}{m} E(1-\cos\theta) = \frac{E^2(1-\cos\theta)}{m + E(1-\cos\theta)}. \tag{6}$$

Equation 1 and 2 reduces to $(p_3 - p_2)^2 = -2mE_3$ and $(p_4 - p_2)^2 = -2mE_4$ respectively. Therefore:

$$\begin{aligned} |\mathcal{M}|^2 = &\quad g^4 \left(\tfrac{1}{(p_4-p_2)^2} + \tfrac{1}{(p_3-p_2)^2}\right)^2 \\ = &\quad \tfrac{g^4}{4m^2}\left(\tfrac{1}{E_3} + \tfrac{1}{E_4}\right)^2 \end{aligned}$$

Plug $E_3$ and $E_4$ from Equation 5 and 6 into this expression, we obtain, after some algebra:

$$|\mathcal{M}|^2 = \frac{g^4}{4m^2} \frac{\left[E(1-\cos\theta) + m\right]^4}{m^2 E^4 (1-\cos\theta)^2}$$

Using Equation 4 again, and apply the ultra-relativistic approximations, we conclude:

$$\begin{aligned} \lim_{v\to c} \left.\frac{d\sigma}{d\Omega}\right|_{\text{lab}} &= \frac{1}{128\pi^2} \frac{|\mathcal{M}|^2 E_3}{mE\left[E(1-\cos\theta) + m\right]} \\ &= \frac{|\mathcal{M}|^2}{128\pi^2} \frac{1}{\left[E(1-\cos\theta)+m\right]^2} \\ &= \boxed{\frac{g^4}{512\pi^2 m^4} \frac{\left[E(1-\cos\theta)+m\right]^2}{E^4(1-\cos\theta)^2}.} \end{aligned}$$

## Problem 5: QED Feynman diagrams [20 points]

Draw the leading-order Feynman diagrams(s) for the following processes:

- Compton scattering - $\gamma e^- \to \gamma e^-$
- Pair annihilation - $e^+ e^- \to \gamma\gamma$
- Light-by-light scattering - $\gamma\gamma \to \gamma\gamma$
- Moller scattering - $e^- e^- \to e^- e^-$
- Bhabha scattering - $e^+ e^- \to e^+ e^-$

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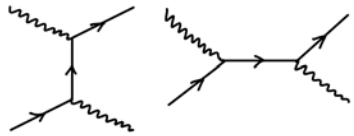
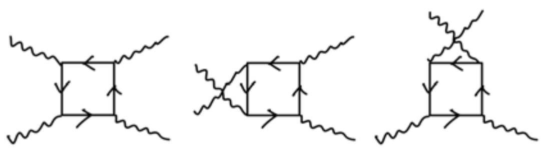
- Compton scattering - $\gamma e^- \rightarrow \gamma e^-$



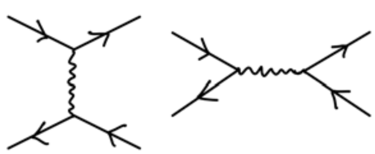
- Pair annihilation - $e^+e^- \rightarrow \gamma\gamma$

- Light-by-light scattering - $\gamma\gamma \rightarrow \gamma\gamma$



- Moller scattering - $e^-e^- \rightarrow e^-e^-$

- Bhabha scattering - $e^+e^- \rightarrow e^+e^-$



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