8.701

Introduction to Nuclear and Particle Physics

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4. QED

4.10 Noether's Theorem

Lagrangian for the free Dirac field

$$\mathcal{L} = i\overline{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\overline{\psi}\psi$$

Unchanged by the global U(1) phase transformation

$$\psi \to \psi' = e^{i\theta}\psi$$

Invariance can be expressed by infinitesimal phase transformation

$$\psi \to \psi' = (1 + i\varepsilon)\psi \quad \text{and} \quad \overline{\psi} \to \overline{\psi}' = (1 - i\varepsilon)\overline{\psi}$$

Lagrangian for the free Dirac field

Changes in the fields and their derivatives are

$$\delta\psi = i\varepsilon\psi, \quad \delta(\partial_\mu\psi) = i\varepsilon(\partial_\mu\psi), \quad \delta\overline{\psi} = -i\varepsilon\overline{\psi} \quad \text{and} \quad \delta(\partial_\mu\overline{\psi}) = -i\varepsilon(\partial_\mu\overline{\psi})$$

Global symmetry implies

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta(\partial_\mu\psi) + \frac{\partial\mathcal{L}}{\partial\overline{\psi}}\delta\overline{\psi} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})}\delta(\partial_\mu\overline{\psi}) = 0,$$

From which we get

$$i\varepsilon\frac{\partial\mathcal{L}}{\partial\psi}\psi + i\varepsilon\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}(\partial_\mu\psi) - i\varepsilon\frac{\partial\mathcal{L}}{\partial\overline{\psi}}\overline{\psi} - i\varepsilon\frac{\partial\mathcal{L}}{\partial(\partial_\mu\overline{\psi})}(\partial_\mu\overline{\psi}) = 0.$$

Lagrangian for the free Dirac field

The terms involving the derivative with respect to $(\partial_\mu \psi)$ can be expressed as

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}(\partial_\mu \psi) = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \psi \right) - \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) \right] \psi.$$

So we find

$$i\varepsilon \left[\frac{\partial \mathcal{L}}{\partial \psi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) \right] \psi + i\varepsilon \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \psi \right) - \{ \overline{\psi} \text{ terms} \} = 0.$$

which leaves us

$$i\varepsilon \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \psi - \overline{\psi} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} \right] = 0.$$

Conserved current

$$j^\mu = (\rho, \mathbf{J}) = -i\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\psi - \overline{\psi}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})}\right)$$

The partial derivatives of $\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi.$

are $$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = i\overline{\psi}\gamma^\mu$$ and $$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} = 0,$$

From which we find the conserved current $j^\mu = (\rho, \mathbf{J}) = \overline{\psi}\gamma^\mu\psi.$