For massive fermions,

$$E\chi_- = -\vec{\sigma}\cdot\vec{p}\,\chi_- - m\xi_+$$

$$E\xi_+ = \vec{\sigma}\cdot\vec{p}\,\xi_+ - m\chi_-$$

or

$$\vec{\sigma}\cdot\hat{p}\,\chi_- = -\chi_- - \frac{m}{E}\xi_+$$

$$\vec{\sigma}\cdot\hat{p}\,\xi_+ = \xi_+ + \frac{m}{E}\chi_-$$

from which, we find the two helicity eigenstates

$$\chi_- + \frac{m}{2E}\xi_+ \quad \& \quad \xi_+ - \frac{m}{2E}\chi_-$$

Note: the "wrong-sign" helicity is suppressed by a factor m/E; Gauge fields preserve fermion chiralities in their couplings.

$$\vec{\sigma}\cdot\hat{p}\left(\xi_+ + \frac{m}{2E}\chi_-\right) = \left(\xi_+ + \frac{m}{2E}\chi_-\right)$$

transformation

$\bar{\Psi}[i\gamma^\mu p_\mu - m]\Psi = 0$

$[i\gamma^\mu \frac{\partial}{\partial x^\mu} - m]\Psi(x) = 0$ — ① $x' = \underbrace{\Lambda}_{= 4\times4 \text{ in } x \text{ space}} x$ $p' = \Lambda p$

$[i\gamma^\mu \frac{\partial}{\partial x'^\mu} - m]\Psi'(x') = 0$ — ②a

$\{i\gamma^\mu \Lambda \frac{\partial}{\partial x^\mu} - m]S\Psi(x)$ $\underbrace{S}_{= 4\times4 \text{ matrix in spinor space}}$ acts only on spinor

S^{-1} $[i\underline{S^{-1}\gamma^\mu\Lambda}\frac{\partial}{\partial x^\mu}\underline{S} - m]\Psi(x) = 0$ ②b   not on x or p

thus commute with Λ

also $\bar{\Psi}' = \bar{\Psi}S^{-1}$

①/②b $\gamma^\mu = S^{-1}\gamma^\nu\Lambda_\nu^{\ \mu}S$ $\Lambda_\nu^{\ \mu}\gamma^\nu = S^{-1}\gamma^\mu S$

$S\gamma^\mu S^{-1} = \gamma^\nu\Lambda_\nu^{\ \mu}$

γ^μ: invariant under transformation Λ

then $\Psi' = S\Psi$

then $S^{-1}\gamma^\mu S = \Lambda_\nu^{\ \mu}\gamma^\nu$ defines $\underline{\underline{S}}$

Since γ^μ is not changed $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

$\bar{\Psi}'\Psi' = \bar{\Psi}S^{-1}S\Psi = \bar{\Psi}\Psi$ scalar

$\bar{\Psi}'\gamma^\mu\Psi' = \bar{\Psi}S^{-1}\gamma^\mu S\Psi = \Lambda\bar{\Psi}\gamma^\mu\Psi$ vector

$\bar{\Psi}'\gamma_5\gamma^\mu\Psi' = \bar{\Psi}S^{-1}\gamma_5\gamma^\mu S\Psi = -\Lambda\bar{\Psi}\gamma_5\gamma^\mu\Psi$ Axial vector

Parity

$$i\gamma^\mu \frac{\partial \psi(x)}{\partial x^\mu} - m\psi(x) = 0$$

$$\Rightarrow \quad i\gamma^\mu \frac{\partial \psi'(x')}{\partial x'^\mu} - m\psi'(x') = 0$$

$$x' = \Lambda x$$

$$\psi'(x') = S\,\psi(x)$$

$$\Rightarrow \quad i\gamma^\mu \frac{\partial S\psi(x)}{\partial \Lambda x^\mu} - m\, S\psi(x) = 0$$

$$i\, S^{-1}\gamma^\mu S \frac{\partial \psi(x)}{\Lambda\, \partial x^\mu} - m\,\psi(x) = 0$$

$$\gamma^\mu$$

$$\therefore \quad S^{-1}\gamma^\mu S = \Lambda \gamma^\mu = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} \gamma^\mu$$

$$\therefore \quad S_p = \gamma^0$$

$$\therefore \quad \psi'_{1,2} = \psi_{1,2} \qquad \psi'_{3,4} = -\psi_{3,4}$$

$$\mathbb{P}_{e^-} = -\,\mathbb{P}_{e^+}$$

$$P_{f\bar{f}} = (-1)^{l+1} \qquad P_c = (-1)^{L+S}$$

28)

Alternative method: Parity

$$t \to t \quad , \quad \vec{x} \to -\vec{x} \quad , \quad \psi(t,\vec{x}) \to \psi^P(t,-\vec{x})$$

$$\therefore \left[i\gamma^0 \frac{\partial}{\partial t} + i\vec{\gamma}\cdot(-\vec{\nabla}) \right] \psi^P(t,-\vec{x}) - m\psi^P(t,-\vec{x}) = 0$$

Since $\gamma^0 \vec{\gamma} \gamma^0 = -\vec{\gamma}$, so

$$\left(i\gamma^0 \frac{\partial}{\partial t} + i\gamma^0 \vec{\gamma} \gamma^0 \cdot \vec{\nabla} \right) \psi^P(t,-\vec{x}) - m\psi^P(t,-\vec{x}) = 0$$

$\gamma^0 \times$

$$\left(i\gamma^0 \frac{\partial}{\partial t} + i\vec{\gamma}\cdot\vec{\nabla} \right) \gamma^0 \psi^P(t,-\vec{x}) - m\gamma^0 \psi^P(t,-\vec{x}) = 0$$

Thus $\gamma^0 \psi^P(t,-\vec{x}) = \psi(t,\vec{x})$

or $\psi^P(t,-\vec{x}) = \gamma^0 \psi(t,\vec{x})$

Note in γ_5-diagonal representation $\gamma^0 \psi(t,\vec{x}) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \xi_+ \\ \chi_- \end{pmatrix} e^{-i \ldots}$

$$\therefore \psi^P(t,-\vec{x}) = \begin{pmatrix} -\chi_- \\ -\xi_+ \end{pmatrix} e^{-ip_\mu x^\mu}$$

i.e. interchange of L-H and R-H chiralities!! In particular, a parity invariance theory must have both RH & LH chiralities equally!! Weak Interaction has only LH, so violate parity max.

In γ^0 diagonal scheme,

$$\gamma^0 \psi = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \psi \qquad \psi = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \qquad \begin{matrix} u_{1,2} & E>0 \\ u_{3,4} & E<0 \end{matrix}$$

so parity $= +1$ for $u_{1,2}$ $E>0$ particles

$= -1$ $\quad u_{3,4}$ $E<0$ anti-..

29)

Gauge Theories

General Relativity
(Theory of gravitation)

Electroweak Theory
(QED + Weak Interactions)

Quantum Chromodynamics
(Strong interactions between quarks)

All of them based on the principle of
Local Gauge Invariance

All forces are carried by

massless "bosons"
(gravitons, photons, gluons)

with the exception of the
weak interactions
based on the principle of

"Spontaneously Broken"
Local Gauge Invariance

QED

Gauge theory of electrons interacting with photons (quanta of the electromagnetic field).

In a theory with only electrons the quantum-mechanical description of an electron involves a complex field $\psi(x)$

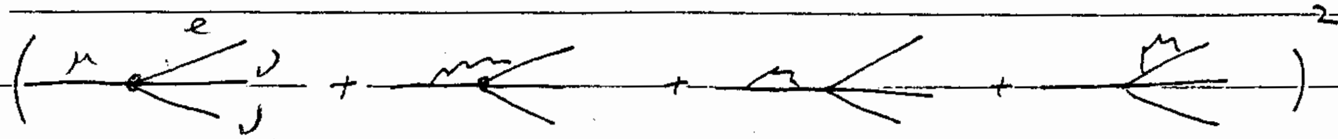
$$x:\ x^{\mu} = (t, \vec{x});\ \mu = 0, 1, 2, 3$$

Probability of finding an electron at "point" x_μ

$$|\psi(x)|^2 = \overline{\psi}^*(x)\ \psi(x)$$

Why Gauge Th.

$M_0 \quad + \quad M_v$



$M_s \quad + \quad M_h$

soft hard

$|M_h|^2 \to$ B.R. $(\mu \to e\nu\nu\gamma,\ E_\gamma > 10\,\mathrm{MeV}) = 1.2\times10^{-2} \longleftrightarrow 1.3\pm0.1\times10^{-2}$

$|M_s|^2 \to$ soft γ $\underbrace{\propto 2\alpha \ln M_\mu/E_{\gamma\,\min}}_{E_{\gamma,\min}=0} \to \cancel{\infty}$

$+ \quad \underbrace{M_0^2 + 2\,\mathrm{Re}|M_0^* M_v|}_{\alpha} + \underbrace{|M_v|^2}_{\alpha^2}$

$\cancel{\infty}$ + finite

Renormalizable Gauge fu
to any order !

$\therefore\ \Sigma = \text{finite} = 1 + (\alpha/2\pi)(\pi^2 - 25/4)$

$= 1 + 4.2\times10^{-3}$

EM $U(1)$ local Gauge transformation

$$\psi(x) \to e^{i\alpha(x)}\psi(x) \; ; \; \bar{\psi}(x) \Rightarrow e^{-i\alpha(x)}\bar{\psi}(x)$$

α continuous real # : Abelian gp $U(1)$

$$U(\alpha_1)U(\alpha_2) = U(\alpha_2)U(\alpha_1)$$

$$(\not{p} - m)\psi = 0 \;\text{①} \to \left(\frac{d}{dx^\mu} - ie A_\mu\right) \Leftarrow D_\mu \qquad \left(d/dx^\mu - ieA_\mu\right)$$

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi = 0$$

$q = -e$ for e^-!

Free e without field is not Gauge Invariant!

$$\partial_\mu\psi \to e^{i\alpha(x)}\partial_\mu\psi + i\,e^{i\alpha(x)}\psi\,\partial_\mu\alpha$$

Define covariant momentum include the field A_μ

$$D_\mu\psi \to e^{-i\alpha} D_\mu\psi \; ; \; D_\mu = \partial_\mu - ieA_\mu$$

where $A_\mu \Rightarrow A_\mu + \frac{1}{e}\partial_\mu\alpha$

Gauge invariant!

$$\mathcal{L} = i\bar{\psi}\gamma^\mu D_\mu\psi - m\bar{\psi}\psi$$

$$= \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + e\bar{\psi}\gamma^\mu\psi A_\mu$$

potential term.

Charge conjugation $\quad e^- \to e^+$, $\quad t \to t$, $\quad \vec{x} \to \vec{x}$

e^- with charge $q = -e$

$$0 = (\not{p} - m)\, u(P) \Rightarrow \left[\gamma^\mu \left(i \frac{\partial}{\partial x^\mu} + e A_\mu\right) - m\right] \psi = 0 \quad \text{—①}$$

$$-e\gamma^\mu A_\mu = \gamma^0$$

$e^+ \quad q = +e$

$$\left[\gamma^\mu \left(i \frac{\partial}{\partial x^\mu} \; e A_\mu\right) - m\right] \psi_c = 0 \quad \text{—②}$$

$$\gamma^2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

How to relate ψ_c with ψ ?

$$\gamma_{1,3}^* = \gamma_{1,3}$$

$$\text{①}^* \quad \left[-\gamma^{\mu *}(i\partial_\mu - e A_\mu) - m\right] \underset{\gamma^2\gamma^2}{} \psi^* = 0 \qquad \gamma_0^* = \gamma_0, \; \gamma^{2*} = -\gamma_2$$

$$(\gamma^2)^2 = I$$

move γ^2 to left

$$\gamma^2\left[\gamma^\mu(i\partial_\mu - e A_\mu) - m\right] \gamma^2 \psi^* = 0 \quad \text{—③}$$

Compare ② & ③ $\qquad \psi_c = i\gamma^2 \psi^*(t, \vec{x})$

↳ phase

$$(\psi)^c = \psi^c = i\gamma^2\psi^* = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} \xi_+^* \\ \chi_-^* \end{pmatrix} = \begin{pmatrix} i\sigma_2 \chi_-^* \\ -i\sigma_2 \xi_+^* \end{pmatrix}$$

$$(\psi_R)^c = (\psi^c)_L \qquad (\psi_L)^c = (\psi^c)_R$$

Weak Interaction violates C maximumly.

CP transformation

$$(\psi_L)^c = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \chi_-^* \end{pmatrix} = \begin{pmatrix} i\sigma^2 \chi_-^* \\ 0 \end{pmatrix}$$

$$\therefore (\psi_L)^{CP} = \gamma^0 (\psi_L)^c = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} i\sigma^2 \chi_-^* \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -i\sigma^2 \chi_-^* \end{pmatrix}$$

i.e. $(\psi_L)^{CP} = (\psi^{CP})_L$ $\qquad \nu_L \qquad \bar{\nu}_L = \bar{\nu}_R$

Chiral $\quad$ helicity

Having only L-H spinor can be CP invariant, but not separately C or P invariant!!

Now $m_\nu \neq 0$

- Majorana spinor: $\psi_M^c = \psi_M$

i.e. $$\begin{pmatrix} \xi_+ \\ \chi_- \end{pmatrix} = i\gamma^2 \begin{pmatrix} \xi_+^* \\ \chi_-^* \end{pmatrix} = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} \xi_+^* \\ \chi_-^* \end{pmatrix}$$

$$\therefore i\sigma^2 \chi_-^* = \xi_+ \quad , \quad -i\sigma^2 \xi_+^* = \chi_-$$

e.g. given ξ_+, $\psi_R = \begin{pmatrix} \xi_+ \\ 0 \end{pmatrix}$ and construct

$$\psi_M = \begin{pmatrix} \xi_+ \\ -i\sigma^2 \xi_+^* \end{pmatrix} = \psi_R + (\psi_R)^c = \psi_M^c$$