HONG LIU: Good. So before we start, do you have any questions on what we did before?

OK. So we have discussed that a black hole can be considered as a similar object. And that, actually, black holes obey the standard law of thermodynamics. And now we also mentioned that it's a natural question to ask whether the black hole should we be considered as an ordinary quantum mechanical object. And so this is related to the issue of information paradox.

So by this regard, this information paradox, which I think now has a very good-- in principle has a reasonable resolution, which we will talk about later. But if we do treat the black hole as an ordinary quantum mechanical object, then they actually are very important implications.

So one implication is called the holographic principle.

So, the holographic principle by itself is, in some sense, a rather heuristic concept at the moment. And I will also only give you a very heuristic discussion. And it's possible to make it a little bit more rigorous. But we will do it a rather heuristic level, just to give you a feeling that there is something profound going on in quantum gravity because of black holes.

So let's consider an isolated system of mass. Say E and, say, entropy as zero, say in asymptotic flat spacetime.

And then we also denote A as the area of the smallest sphere that closes the system. And also let me introduce-- let MA be the mass of the black hole, of the same horizon area. Yes. Black hole with horizon area A. OK?

So in GR, anything which is sufficiently massive will eventually collapse to form a black hole. Because of that-- so we must have E is smaller than MA, So we must have E-- so whatever energy you can put in this region surrounded by area A must be smaller than the mass of a black hole which you can put in that region. OK? Because otherwise, if we have more mass, then it will already be a black hole.

AUDIENCE: Should that be equal?

HONG LIU: Hm?

AUDIENCE: Should it be equal?

HONG LIU: Yeah. Say equal.

So now let's suppose E is smaller than MA. So we can now-- so suppose E is smaller than MA and now we can add this amount of energy, say MA minus E amount of energy, to the system.

OK. Suppose we can do that and keep A fixed. Suppose we can do that and keep A fixed. And then this will reach a black hole mass. Then the final system will have a black hole, will have the energy given by MA. And that object must be a black hole because that's the black hole with the same horizon area.

Because this process also tells you that the black hole entropy must be greater than the S0, the initial entropy of this system, plus the S prime-- S prime is the entropy of added energy.

OK. So just because of the second law of thermodynamics, your entropy should not decrease. So the final-- so you add this amount of energy so to MA, so then you will precisely form a black hole over the mass MA, and with the horizon area A. But this black hole entropy should be greater than its zero plus the entropy you added from the second law of thermodynamics.

So then that means your initial entropy must be smaller than the black hole entropy. Must be smaller than A divided by 4 h bar G newton.

So this argument tells you that the maximal entropy inside a region bounded by area A is given by--

OK. Yes?

AUDIENCE: So will keeping A fixed add entropy to the system?

HONG LIU: Yeah.

AUDIENCE: So in that case, why don't we have such a term?

HONG LIU: Sorry?

AUDIENCE: So you said we want to keep A fixed.

HONG LIU: Yes.

AUDIENCE: Will that add extra entropy to the system?

HONG LIU: Yes.

AUDIENCE: So according to the second law of thermodynamics, why will we claim that the entropy of the hole is greater than or equal than S0 plus S--

HONG LIU: Yeah.

AUDIENCE: Oh. OK. It's add the energy.

HONG LIU: Yeah. Because it-- Yeah. This is the energy of added energy.

AUDIENCE: How about the extra energy that might be introduced by keeping A fixed? Because it's not--

HONG LIU: Yeah. Whatever it is. Whatever it is. Suppose you can keep A fixed. Suppose you can keep A fixed. Then whatever things you added to that-- that total entropy-- must be smaller than the [INAUDIBLE] because black hole is the final state.

AUDIENCE: You said it's an isolated system, so with that, how can you make sure that A is fixed?

HONG LIU: Hm?

AUDIENCE: Why is an isolated system--

HONG LIU: Oh. Yeah. I'm just saying, we are doing a thought experiment. Imagine you can do this by keeping A fixed. Yeah, just say you just put stuff in into this region. So that stuff does not come out. Yeah, you just have some region A. Let me do it here. You have some region A. And there is a mass inside here. There's some energy stuff inside here. So this is region A. And then you just keep putting stuff in. Just keep putting stuff in so that those things don't come out A-- don't come out of this region. That's what this means.

AUDIENCE: Then A is not the smallest sphere?

HONG LIU: Hm?

AUDIENCE: Then A is not the smallest sphere?

HONG LIU: A is what?

AUDIENCE: Not the smallest sphere that can enclose the system. So it's not--

HONG LIU: No. A is not the smallest sphere in the system. A is just-- maybe I should call this-- A is the area of a sphere. Yeah. Sorry. Yeah. Thanks. So just enough A to be a sphere that encloses the system. Yeah. Should not put the smallest sphere for some reason. Yeah. No, no, no. A should not be the smallest. Just-- A is any sphere. Just some sphere which encloses the region. Yes?

AUDIENCE: So it seems that this argument implicitly assumed that in order to add entropy, that sort of corresponds to adding some amount of energy. But is it possible to add entropy to the system without adding energy to the system?

HONG LIU: Oh, you can. You can. They just make it even more. Whatever it is, this is the final state. So the base assumption of this is that if you can put stuff inside this region, whatever is inside, the sum is, the maximal stuff you can put inside the region is a black hole with the horizon area.

AUDIENCE: Well, I guess what I'm confused about is that it seems, that-- I certainly believe when you said, you can't have a certain amount of energy in an area. But I'm not sure why you can't have a certain amount of entropy in an area if energy and entropy are not--

HONG LIU: No. We're not making that assumption. The entropy is just a consequence of that. This is the final state entropy. And the final state entropy is always larger than the entropy of your initial state.

AUDIENCE: OK. Sure. Fine.

HONG LIU: Yeah. Yeah. Entropy is derived. This equation is not assumption. It's derived.

AUDIENCE: Fine. OK.

HONG LIU: The assumption is that the black hole is the maximal-- is the most massive object you can put inside this region.

AUDIENCE: Sure. I see.

HONG LIU: Yeah.

AUDIENCE: But you can't have a less massive object with more than that entropy.

HONG LIU: Hm?

AUDIENCE: You can't have a less massive object with entropy that is more than that value. Inside the same region.

HONG LIU: This is ruled out by this-- once you get this equation, then you don't ask about the mass. It just-- any entropy must be smaller than this value. Any other questions?

AUDIENCE: How do we keep track of what the area of [INAUDIBLE]

HONG LIU: Right. So that's why I say this is a little bit heuristic argument. And that's why I say this is a heuristic argument. Yeah. Yeah, indeed, you have to be a little bit more careful about that. Yeah but say if you-- the key thing is about the area. And then this other one cautions one maybe to have to think about it. When you include the quantum corrections et cetera-- the back reaction, et cetera. But the key thing is the area of the black hole. It must be proportionate to the area.

AUDIENCE: Well. [INAUDIBLE] for some function that the area of the horizon with the black hole is only based on the mass.

HONG LIU: Yeah. Yeah. Yeah. Yeah. So let me say again, there's two natures of this argument. The first nature of the argument is, the maximal energy you can put inside the region is a black hole whose horizon area is given by the area of this region.

AUDIENCE: This is an assumption.

HONG LIU: Yeah. This is assumption. Yeah, because we know, in the GR, any sufficiently massive thing will collapse to form a black hole.

AUDIENCE: Mmhmm.

HONG LIU: Yeah. And if you have a more mass, then the horizon area would be outside. And then you will be outside this area. You can no longer keep this A fixed. So that's the first assumption. So we said that assumption-- that assumption then just follows from the second law of thermodynamics. Because this is whatever-- the entropy of the matter inside-- then this is whatever the initial entropy of the energy-- you need to put it in to make it into a black hole. And then that must be smaller than the entropy of the final state, which is the entropy of the black hole.

AUDIENCE: But my question is that by keeping A fixed, you might introduce extra entropy into the system.

HONG LIU: Hm? Yeah. It doesn't matter. As far as the black hole is the final entropy state. What you're saying is that there might be some other things here. It's fine. That does not change the direction of this--

AUDIENCE: The entropy might be negative. I'm not sure-- just--

HONG LIU: No. No. Whatever it is-- this is my initial. So that's why this is an inequality here. This is not the equality. Inequality. And we're just using the second law of thermodynamics. This is your initial state entropy. This is my final state entropy. This must be greater than that. We're not asking the specific process. We're not asking the specific process. Yeah. Just, the final state must be greater than the initial state. Good? OK.

So now let's see what we can try to go a little bit further from here. So now let's recall. So now we really treat the black hole as a quantum statistical object. So now recall the definition of entropy in quantum statistical physics.

OK. So if you say I have some system, and the state of that system is described by some density operator, or density matrix, and then that the entropy of that system is just given by this formula. In particular, for a system with n-dimensional Hilbert space.

OK. So this is the n-dimensional Hilbert space. This is a dimension of the Hilbert space. Let S max is log N. So the maximum possible entropy you can have is log N.

OK. Is this fact familiar to you? So you can convince yourself the maximal entropy is realized when role is just given by 1/n times the identity matrix. And so is the density matrix which has low information. Because every state plays the same role. And this state-- then you can plug in here and give your log N. And the maximum entropy means you have minimal information. And that is the state with minimal information. And you can also just prove yourself using some algebra. But this is the physical argument for this.

So now, if you compare with these two formula, then we can draw a conclusion, is that-- so let me add a word called effective-- effective dimension-- of the Hilbert space for a system inside a region of area A is bounded by log N must be smaller than A h bar GN.

So that tells you-- so whatever you can put inside of this region of area A, that's the maximal entropy you can have. So if you have N degrees of freedom, then the maximal entropy you can have for this N degrees of freedom is log N. Then that tells you that log N must be smaller than this. OK? Smaller or equal to that. And so let me write this in terms of the-- so this can also, 4 h bar N is also Planck length square. So it's also four times a Planck length square.

AUDIENCE: So for this part, really Hilbert space really can be even infinite.

HONG LIU: Yeah. And let me say that. So that's why we need "effective." So now let me make some remarks. Let me make some remarks.

So zero remarks. It says, for system of n spins say, n spins half particle-- so when I say spins, I always means spins half-- then the dimension of the Hilbert space is 2 to the power n. Just to remind you. But the dimension of h-- the Hilbert space-- so normally I just group h to denote the Hilbert space-- for a single harmonic oscillator. So the simplest system is spin. Then the next simplest system is essentially a harmonic oscillator. And even for a single harmonic oscillator, then the dimension of h is infinite.

OK. And then you ask what I'm talking about here. Because here I say the Hilbert space should be smaller than that. But. So even though, even for a single harmonic oscillator the Hilbert space is infinite, but for a quantum mechanical system, for a quantum system with a finite number of degrees of freedom-- by finite number of degrees of freedom, for example, you can interpret this, for example, as a finite number of harmonic oscillators, OK-- the dimension of the Hilbert space below some finite energy scale. Energy is always finite. It's always finite.

And so that's why I use the word "effective" there. And because normally we all consider the finite energy, because the system will have finite energy, and then you have, effectively, a finite dimension of Hilbert space. OK?

So this equation is not that crazy. So I mean typical-- so actually, in typical physical systems, even for harmonic oscillators, both for spin or harmonic oscillators, essentially, number of degrees of freedom is of all the log N. Yeah, N is the effective dimension of your Hilbert space. Yeah, this is obvious from the spin. This is also obvious if you have a harmonic oscillator. So if each harmonic oscillator is finitely excited. And again, roughly the log N is the number of harmonic oscillators.

So because of this we can also write this equation as the number of degrees of freedom of any quantum gravity system inside the region A can be-- should be smaller than-- should be bounded by the area of that region.

So this bond is certainly violated in non-gravitational systems because in non-gravitational systems whose number of degrees of freedom and thus, the log N, the log of your dimension of your Hilbert space, is proportional to the volume rather than area. So when the reading is big enough, then you are guaranteed to violate this bound. Because, of course, the volume grows faster than the area if you have a sufficiently larger region.

So for example, consider just a lattice of spins-- so a three dimensional lattice of spins-- of lattice spacing a-- say of lattice spacing a-- then the total number of spins that is given by volume divided by A cubed.

So this is essentially the number of degrees of freedom. And then this is given by A, the area, divided by a squared, times L divided by a. And this can be much, much greater than a divided by Planck square, Planck length square, for L sufficiently large. They are large enough.

So if you have a long gravitational system, then this boundary can be easily violated. In particular, this equation can also be violated because the dimension of the Hilbert space that in this case would 2 to the power a cubed, and similarly-- so a-- so the maximal entropy would be, say, v divided by a cubed log 2, because it's a spin 2 system, spin half system. So this can be larger than the entropy of a black hole.

So it's key that we can see the gravitational systems. Only in gravitational systems you have this remarkable feature that if you-- things will collapse. Things will collapse. And then you will have this upper limit provided by the black hole. Yes?

AUDIENCE: So in the real world, every system's a gravitational system.

HONG LIU: Yeah.

AUDIENCE: So this is just-- in some sense, this is not actually correct in the real world.

HONG LIU: Yeah, but you need this L to be not enough.

AUDIENCE: I see.

HONG LIU: Yeah. Yes?

AUDIENCE: So, right. In the real world, this bound has to be obeyed by systems. So does that put some bounds on QFT UV completions?

HONG LIU: Hm?

AUDIENCE: So, in the real world this bound has to be obeyed by everything.

HONG LIU: Yeah.

AUDIENCE: So does that put some kind of restrictions on QFT UV completion? Because we usually say it's infinite density of degrees of freedom.

HONG LIU: No, because UV completion of QFT is provided by gravity.

AUDIENCE: For any QFT. I just mean, like, a scalar fuel.

HONG LIU: Then that's not the theory of gravity.

AUDIENCE: Right, but if you have a scalar fuel in space time, which has GR in it, is otherwise like, flat?

HONG LIU: Yeah. Then the GR-- then the gravity will provide the UV completion on that scalar field.

AUDIENCE: Sure.

HONG LIU: Yeah. Right. I was going to say something else, but I forgot.

AUDIENCE: [LAUGHTER]

HONG LIU: Haha.

AUDIENCE: What about the ones that are to be complete?

HONG LIU: Hm?

AUDIENCE: What about the theories that are to be complete at least, formally?

HONG LIU: Uh. Sorry, say it again?

AUDIENCE: The QFTs that are to be complete.

HONG LIU: Yeah. They are limited by gravity.

AUDIENCE: OK.

HONG LIU: Yeah. The gravity will still provide something. Because a QFT have infinite number of degrees of freedom. Yeah.

So as an example of this, for example you can think of a classroom, say, full of photons, and then you can calculate entropy. Again, that entropy will be proportional to the volume. So if you make the classroom big enough, then that will violate this bound. But it's a simple exercise for you to do yourself, that if you put the classroom at a finite temperature of photons, then when the room is big enough, there's always a radius that will collapse to form a black hole before this bound is violated.

AUDIENCE: [LAUGHTER]

HONG LIU: OK. I think maybe I have this formula in the p-set. I forgot whether I have it or not. Yeah, you can do this simple exercise yourself. Yeah just imagine you have a box of photons at any non-zero temperature t, and then when you put the box big enough, then at a certain point then this whole box of photons will collapse to form a black hole. And that's within to make this bound to be satisfied.

So what this means-- that if you have gravity, the quantum gravity leads to a huge reduction of number of degrees of freedom. It's really bounded by your area.

AUDIENCE: Now, in previous argument [INAUDIBLE], you have a conformal field theory. And so this other [INAUDIBLE] we don't need to have [INAUDIBLE].

HONG LIU: No. Yeah. As I was answering Pavel before, if you have a conformal field theory just by itself, then there's no bound.

AUDIENCE: Yes.

HONG LIU: Then there's no bound.

AUDIENCE: Oh.

HONG LIU: But if you couple this conformal field theory to gravity, then at sufficiently high energy, and then the degrees of freedom of this conformal field theory must be reduced.

AUDIENCE: No, but, if you just had a conformal field theory, then there's no bound.

HONG LIU: No bound. Yeah. Yeah, but you can ask what happens if we couple a conformal field theory to gravity. Not even conformal field theory, just any quantum field theory, because any quantum field theory has an infinite number of degrees of freedom, and the number of degrees of freedom is also proportional to the volume. And if you couple it to gravity, then this reduction must happen. This reduction must happen. OK.

So now if we take a small leap of faith-- so we already have taken some leaps of faith, but let's take one more stab, or leap of faith-- then you can formulate something called the holographic principle.

So given that the number of degrees of freedom inside the region is always bounded by the area, and this area actually has a very nice way, it's area in the unit of a Planck lens-- it looks like we have a minimal lens-- minimal area which is a Planck's square lens-- and just the area divided by that gives you the total number of degrees of freedom. OK.

So if you try to extrapolate a little bit further, then we have the so-called holographic principle. It says, in quantum gravity, a region of boundary area A can be fully described-- can be fully described by no more than-- so essentially it's just A h bar hN equals A [INAUDIBLE] squared degrees freedom. In other words, one degree of freedom per Planck area.

So this is a rough formulation of the holographic principle. So what we just said-- it just highlights a point we made earlier-- is that even though naively, from dimensional analysis, that the quantum gravity is only operating, say, at the Planck scale. Say, at the Planck mass or [INAUDIBLE], et cetera. And we see that the black hole actually brings the quantum gravity really need to macroscopic level. OK. Because this is, really macroscopic statement can apply to regions of arbitrary sides. But somehow, they are limited by constraints of quantum gravity. OK. So this is a remarkable manifestation of the quantum effect.

Any questions on this? Yes?

AUDIENCE: What's so special about the dimensional differences?

HONG LIU: Oh. Yeah. Yeah. This we can generalize to any dimension. There's always core dimension two. So if you live in five dimensions, then becomes three dimensions-- yeah, it's always co-dimension y in the spatial region. Because the black hole-- just area of the black hole is always co-dimension y in terms of spatial. Yeah.

AUDIENCE: Well, can't it be just [INAUDIBLE] sphere, and then it's [INAUDIBLE]

HONG LIU: Yeah. 10-dimensional space, then naively, this would be constrained by eight-dimensional surface. Yeah. Yeah, just, no matter what dimension, it's always constrained by the surface, not constrained by the volume. No matter what dimension. Other questions?

AUDIENCE: I just have a question about [INAUDIBLE].

HONG LIU: Yeah?

AUDIENCE: So it seems like we are constraining a system [INAUDIBLE] theory that its number of states for a given energy must grow slower somehow because we must be lower than some finite energy.

HONG LIU: Yeah.

AUDIENCE: The number of [INAUDIBLE] must be below certain--

HONG LIU: Yeah. Just say-- essentially, what we said earlier, is if you are in the theory of quantum gravity, there's a cutoff on the maximal energy you can have. And there's a cutoff also on the maximal energy you can have, because otherwise you form a black hole, and then that cutoff will translate into the state of the [INAUDIBLE] of entropy and into the number of degrees of freedom.

AUDIENCE: Yes, I mean--

HONG LIU: Because much easier-- yeah, yeah.

AUDIENCE: I mean, for example, we have-- first we start with a simple series, and maybe, for example, we add, for example, standard model, then we add super symmetry, then-- energy spectrum-- the limiting energy that is there, but the number of Hilbert space state is more, and we can keep doing things like that.

HONG LIU: Yeah.

AUDIENCE: But we can't. Because it must be below these values.

HONG LIU: Yeah. But you see, there is something about the volume here. You have to talk about within some region. Of course, abstractly, you can put some degree-- yeah, maybe I'm not getting your question.

AUDIENCE: How do you say-- naively, we think, we can just, for example, say if the energy boundary is here, and below this, how many states we could have?

HONG LIU: No, no, no. This is not the direction in terms of the entropy-- energy bound. This is, in the end, is the bound on the [INAUDIBLE] degrees of freedom you can put inside a region. If you put too many degrees of freedom inside the region, then you need to enlarge the region. Yeah. Just say, look at the number of degrees of freedom is incompatible. Something like that.

AUDIENCE: OK. I have a question regarding [INAUDIBLE].

HONG LIU: Yeah.

AUDIENCE: I still don't understand. Shouldn't an isolated system mean also included the machinery to do these things?

HONG LIU: No. I'm just saying you start with the isolated system. Then you can add something to that isolated system.

AUDIENCE: What about if we take matters out of the black hole, these [INAUDIBLE]?

HONG LIU: That thing, that isolated system, does not have to be a black hole. It can be anything. It just can be anything.

AUDIENCE: So if we instead of putting energy inside, we take energy out, entropy will decrease?

HONG LIU: Yeah. Just take entropy out. Yeah. Then this bound will be even more satisfied.

[LAUGHTER]

AUDIENCE: I have a question there, that the degree of freedom-- I mean log of the degree of freedom--

HONG LIU: No. No. N is the dimension of the Hilbert space. N is the dimension of the Hilbert space. That's why I called it N, here.

AUDIENCE: Yes.

HONG LIU: N, the dimension of the Hilbert space.

AUDIENCE: But-- the degree of freedom is--

HONG LIU: Typically, the number of-- the dimension of the Hilbert space is exponential of the lambda of degrees of freedom.

AUDIENCE: Oh. OK.

HONG LIU: Just look at here.

AUDIENCE: Oh. I see.

HONG LIU: Just look at here. Yeah. Any other questions? OK.

AUDIENCE: If we have a quantum field theory, it has a certain vacuum energy, right? And then if you now consider that to be the vacuum energy and you expand it at some point, it will actually collapse into a black hole? So if you take the vacuum energy to be that predicted by QFT, then for the universe, it would not work. Even theoretically-- I know the [INAUDIBLE] constant is like 120 orders less, but.

HONG LIU: Yeah, that's a little bit tricky because-- yeah, so the vacuum energy will make it into, say, depending on whether the vacuum energy is positive or negative will make it into de Sitter, anti-de Sitter, then, yeah, then you try to see whether-- it depends on details.

AUDIENCE: But if you go big enough, it's going to collapse, or--

HONG LIU: No. Not necessarily. I'm just saying that you have to formulate a little bit differently. You're talking about a particular state of certain energy to couple to gravity, then you just take that state coupled to gravity to see what you get. Yeah, then ask questions from there. Yeah. Good.

So now let's move to a new topic. Still, it's in the preparation to go to the duality but we need a lot of pieces of important intuition in order to fully appreciate the duality. And in this piece of intuition is Large N theories. And let me call it Large N matrix theories.

So now we try to look at the hint for holographic duality from the field theory side. So so far, we have been talking about the hint for holographic from the gravity side, and now we look at it from the field theory side, to see, actually from field theory side, there also some hints to relate a long gravity system to a gravity system. OK.

So let's start with QCD. So QCD is a SU(3) gauge theory plus quarks. So let me just write down it's Lagrangian.

So this F describes the gluon fields and this side describes the quarks. OK. So this F is built out of the standard partial mu A mu minus partial mu A mu minus A mu A mu, and A mu-- each gauge field is a 3 by 3 Hermitian matrix.

So these are 3 times 3 Hermitian matrix. So this is using the matrix notation, so F is also a matrix. So there's a trace here. OK. Hermitian matrix. And then the T to a is just generators of SU(3).

So it doesn't matter if you don't know QCD. It doesn't matter if you have not seen QCD before. The only thing that matters is just to remember that QCD is a series of matrices.

AUDIENCE: [LAUGHTER]

HONG LIU: QCD is a series of 3 by 3 matrices. OK. So the gluon field is 3 by 3 matrices. It's described by matrices. So of course, one of the remarkable things about the QCD-- what's the most remarkable thing about QCD, do you know?

AUDIENCE: [INAUDIBLE].

HONG LIU: Yeah. That's one.

AUDIENCE: [LAUGHTER]

HONG LIU: And what other remarkable thing about the QCD?

AUDIENCE: Soft coupling?

HONG LIU: Hm?

AUDIENCE: Soft coupling.

HONG LIU: Yeah. It's related to that.

AUDIENCE: Confinement, then.

HONG LIU: Yeah, confinement, it's also related to that. It's just-- yeah, one of the most remarkable things about QCD is it's still not solvable.

AUDIENCE: [LAUGHTER]

HONG LIU: And the reason it's not solvable is because when you've got low energies, the coupling becomes strong, then we don't know how to deal with it. OK. So we still don't know how to calculate it from first principle. Of course, we car put it on the computer, but just from the-- in terms of theoretical understanding, it's hard.

So in 1974-- so this is an old problem-- so starting in 1974-- starting in 1971, people already tried to solve it. So in 1974-- so the big problem with of this thing is that in quantum field theory, normally we do expansions. OK. Say we find some starting point, so in physics, that's what we always do. We find the starting point, which can be solved. Then we expand around that starting point. And that starting point is often a free theory. And then it may expand upon the free theory.

So the difficult thing about this theory is at first, it's hard, when you go to low energies-- first it's hard to find the starting point. And the second is that's there's nothing to expand. Because there's no small parameters. OK. There's no small parameters. When you need to expand, you will always need a small parameter.

So in 1974, t' Hooft came up with a very nice idea. He said, this is SU(3). He said, let's just think of taking the number of colors, which is N equal to 3, as a parameter. So even though you realize, he said, there's only three types of colors. But let's imagine we treat it as a free parameter. OK.

So this just implies, say you promote the gauge field by M matrices. And by N Hermitian matrices. OK. He said, let's just treat this in terms of N by N matrices. And then this let's consider that N goes to infinity limit.

So let's consider that N goes to infinity limit. So he reasoned, if N goes to infinity limit, it's sufficiently simple, then maybe we can solve that limit. And then do 1/N expansion. 1/N is a small parameter. Then we can treat 1/N as a small parameter and then do one of the expansions. OK. So that's the rough idea. And then you get a small parameter of 1/N.

So in real life, N will be three. So 1/3. It's not very small, but it's at least smaller than 1. OK, so, this turns out to be a genius idea. Turns out to be a genius idea. But unfortunately, QCD still cannot be solved this way.

AUDIENCE: [LAUGHTER]

HONG LIU: But there's a surprise. But there's a surprise. So, still, QCD, you cannot solve this way. Even N going to infinity limit turned out to be too hard. But there's a surprise. It turns out this 1/N expansion gives you a string theory. OK.

So I will now explain this, OK. I will now explain this. So there's a byproduct of his large N expansion. You said, you find that this actually-- such a gauge theory actually sticks with real string theory.

So the key is that these fields are matrices. OK. So the key reason this emerges is because we are working with a field, which are matrices. So to illustrate this, I will not do the QCD. It's a little bit complicated. I will just do a simple scalar field theory. OK. So I will just do a simple scalar field theory as an illustration.

So let's consider the following theory.

OK. So this g is still coupling constants. But what we do is actually convenient to put it in the front. Of course you can rescale it-- you can rescale it, you put it here, which is the normal place. But I put it in the front. Just like what we normally do also for QCD. So this g is coupling constant which we put in the front.

And then each phi, again, I take it to be N by N in the Hermitian matrix. Phi x, each one is a matrix a, b. So I have two index. So this N by N Hermitian matrix. Yeah. N by N, Hermitian matrix. So which means that phi a b, equal to star equal to phi b a. OK.

So there's a difference between the upper and lower indices. OK. So I can write it more explicit in terms of components. So just to give you, in case some of you are not familiar with this kind of matrix kind of field theory, so let me just also write it in terms of components.

So writing in terms of components, it's 1/2 partial mu, phi a b, then partial mu, phi b a, then plus 1/4 phi a b, phi b c, phi c a, no-- phi c d, phi d a. OK. So this is the just having writing down explicitly using components.

So this L, by definition, is invariant under, in addition to the usual translation rotation symmetries, et cetera, has also internal symmetry-- under UN, global symmetry. OK.

So say phi x goes to U, phi x u dagger, for any u, any constant, is u N matrix. So this series is invariant under-- so this is obvious from here, because of this trace, so this is invariant under this. Hm?

So let me just make some remarks. Before we proceed to talk about the features of this series, let me just make some quick remarks.

Zero. This is a theory of N squared scalar field. OK. So if this is intuitive to you, then this just means you have many, many fields coupled together in some specific way.

And also we don't have to consider Hermitian matrices. To consider Hermitian matrices just for simplicity, one can also consider other types of matrices.

For example, real symmetric matrices. So it's real symmetrical, anti-symmetric matrices. So in those cases, if you consider, say, real matrices, then symmetric or anti-symmetric matrices, in that case, this global symmetry becomes s o N, rather than u n.

So in here, this u is a global symmetry. This u has become and cannot depend on x. You see from here, it cannot depend on x. But you can also-- can also introduce gauge fields to make this u n symmetry local.

So I will not do it now, but this will be important a little bit later. So later, we will indeed make it into a local gauge symmetry. But for now, let we just consider this simple example. Because I don't want to introduce too many fields, et cetera. So any questions regarding the setup of this theory?

AUDIENCE: Yes, can you explain that last point one more time? What do you mean the-- if you-- how does introducing gauge fields make the symmetry local?

HONG LIU: Yeah. You have to take 8324 to know that.

AUDIENCE: [LAUGHTER]

HONG LIU: But the simple way to explain it-- yeah, there's no simple way to explain it.

                  [LAUGHTER]

AUDIENCE: I'll draw the fields, I guess.

HONG LIU: Yeah. Yeah, just say-- let me just explain the statement. If you introduce some gauge fields, then you can make this u to be some arbitrary function of x, rather than a constant. So that's the statement of a local.

AUDIENCE: That's fine.

HONG LIU: Yeah. Just so I can generalize this theory, so that this symmetry becomes local--

AUDIENCE: Ah. I see.

HONG LIU: Becomes a symmetry so that u can have arbitrary x dependence-- can have arbitrary spacetime dependence. So right now, u must be a constant. Any other questions? Good.

                  OK. So now, let's just try to do perturbation theory. Think g squared small. Try to do perturbation theory. OK. So the first thing you do-- of course, you should write down the Feynman rules.

                  Say, first thing, you have to look at the propagator. Look at the propagator. So you can, essentially-- you are all field theory experts, so just by looking at that Lagrangian, then you can immediately write down this propagator. We can just write down x minus y. So whatever propagator-- whatever spacetime dependence, let me just call it x minus y. But also there's the nontrivial index structure.

First there's also g squared. OK. Because there's a prefactor 1/g squared. So when you invert the connection, there's a g squared. But the index structure is at a b, b a. So that a must contract with d and b must be the same as c. So that means we have delta a d and delta b c. OK. Because in the Lagrangian, only those fields are paired. So the propagator must have this structure.

So similarly, we can write down the-- so there is a four particle interaction vertex. So each particle has two index, so this is a b, c d, e f, oh yeah, right, sorry. Let me add one more. And then I can denote this diagramatically just as a b c d, OK. Just denote this diagramatically as the propagator, which is wavy line a b c d. OK.

And now, also, we have this quartic interaction vertices, which is involving four fields. So each field has two index, because a b, so a b c d, f g h. But this is another arbitrary interaction they have contracted. In the interaction, they contract in a specific way. So you get 1/g squared because of [INAUDIBLE] factor g squared. And then you have delta b c, delta d e, delta f g, delta h a. OK, because they have to be contracted. Better to write it as a b, this way. a b. OK.

So a b, b contracted with c, d contracted with e, e contracted with g, h contracted with a, so this gives you a matrix product. So just like that, OK.

AUDIENCE: So it doesn't propagate or anything. It just changes a and b.

HONG LIU: Hm? a b

AUDIENCE: --contracted with d.

HONG LIU: Yeah. Let me just [INAUDIBLE] d c. That's right. OK. So this is a Feynman rule. So now you can start calculating.

So if you have calculated this for a while, then you find this is a little bit tedious.

AUDIENCE: [LAUGHTER]

HONG LIU: With all these indices, it's quite annoying. But t' Hooft was a smart fellow. He said, let me find a way so that I don't have to keep track of all these indices. Then he invented a very nice way. So this is called double line notation.

So first, this propagator a b c d, since that a is contracted with c, c is contracted with b separately, he said, let's just write it as a double line. But since there's a difference between the upper the lower indices, so let's define a direction for this double line. So let's define the direction to go from the upper index to the lower index. Then you will have a b, then you have d, then c b. Because for c, is upper to lower index, to go from c d.

So now you don't have to worry about this thing anymore, because it's automatically taken care of by this notation. Similarly, this one is also automatically taken care of because each one, if you replace it, by the double line notation, then you just make it a, b, then b is equal to c, b equal to it here, c equal to c, d equal to d, and a to a. And then we have g squared, here you have g squared.

So once you do all the double line notation-- so let me also draw the arrow-- once you have double line notation, then all of those annoying deltas, they're automatically taken care of.

AUDIENCE: On the left, that side, [INAUDIBLE]

HONG LIU: Hm?

AUDIENCE: c is on the top of d.

HONG LIU: Sorry, which one?

AUDIENCE: That [INAUDIBLE]

HONG LIU: Here?

AUDIENCE: Yes, c is on the top.

HONG LIU: That--

AUDIENCE: Yeah, if we write [INAUDIBLE]

HONG LIU: Oh. OK. delta b c. That's right. b c. Good. Good. Good.

So now let us give you some examples of using this diagram to calculate things. And also try to calculate the independents. Because eventually we want to take a large N limit. OK. So we want to keep track of the independents.

HONG LIU: Yes?

AUDIENCE: The four point coupling? Does that come from the commutator in the f [INAUDIBLE] in the gauge theory?

HONG LIU: No, I just wrote it down by myself.

AUDIENCE: OK, so it's an additional thing.

HONG LIU: Yeah. Yeah.

AUDIENCE: So [INAUDIBLE]

HONG LIU: It's actually-- yeah. Just arbitrary field theory. Just an example.

OK. So now let's start calculating. You already have the Feynman rule. What are you waiting for?

AUDIENCE: [LAUGHTER]

HONG LIU: Haha. So let's just calculate. So the first thing, let's look at the vacuum energy. So that's the first thing to calculate. So what's the vacuum energy? The vacuum energy is all the diagrams. It's a summation of all of the diagrams without external [INAUDIBLE]. OK.

So the simplest diagram is just with one vertex. OK. So this is the simplest diagram. But now, so if you do lambda 5 4 theory, then that will be the end of the story. OK. You just contract them. But now, the matrix, because we have contracting matrices, so this is like contracting to four matrices. When you're contracting matrices-- but matrices don't commute.

So actually there's a difference. So you should convince yourself there's a difference which I actually connect these two from which I connect these two and these two. OK. Because this corresponding to I connect with neighboring ones. And this one I connect with diagonal ones. And the order actually matters. The order actually matters. Because matrices don't commute.

So if I call this a and the b, and a not equal to b. Because the matrices don't commute.

So now let's do the N counting. So to do the N counting is actually much simpler if we use the double line notation. So let's now use the double line notation. Because all these N, so when you do the contractions, you have all these indices, you sum all those indices-- when you sum all those indices, you will get some factors of N. So that's how N will come into this theory. But double line notation makes the constraint all manifest, so the contraction will be automatic. You don't have to worry about all those deltas.

So let's just do that. Let's draw this four point vertex. For this one, we just contract the neighboring ones. For this one, we do the same thing. We contract the neighboring ones. Yeah. I did not draw very well. Let me--

OK. So this one should be contracted with this one. This one should be contracted with that one. And this one contract with this one, and this one is contracted there. OK. There's some intersection.

So now let's come to the end. So now here is the key. Each loop corresponding to a contracted indices. So for example, in here, this is already a and a. And then if you contract these two, then you sum over this a when you calculate the propagator. And then that gives you a factor of n. So each contract, each closed loop, essentially gives you a factor of n. Just because you contract-- yeah. So there are three loops here. Three contracted loops. So this gives you n cubed.

And if you look at the g squared dependence, so you have two propagators here. You have two propagators. Each propagator gives you a g squared. Then you have one vertex. Each vertex gives you 1/g squared, so you have g squared. OK.

So now let's look at this one. This one, this says follow this line-- oh, you should also draw the arrow, because the arrow should follow. I did not draw the arrow, but arrows should all follow.

The arrow should be consistent. Similarly, here.

So here, if you go around, you find there is only a single contracted line. So this, only a factor of n. Of course, g squared dependency is the same. So you have g squared. So now you already see the difference between these two diagrams. One is proportionate to n, one is proportionate to n cubed.

So a, diagram a, can be drawn on the plane-- we are drawing it on the plane without crossing lines, so we call it a planar diagram. And b cannot be drawn on the plane without crossing lines, so we call it a non-planar diagram.

So let's do one more step. So let's do one more diagram.

So let's do one more diagram.

So next order, you just get two, next order [INAUDIBLE] diagram without [INAUDIBLE] you have two vertices. OK. So you can connect this way. So this is one diagram. You can also have a non-planar diagram-- yeah, let me just draw the straight line, rather than the wavy line, just to save trouble. You can, for example, you can draw like this, so this is a non-planar diagram.

So now I'll give you an exercise today when you go home. Turn those into double line diagrams. Then you should find yourself that this is n to the power of 4. Yeah. Maybe I should do this. Yeah, let me do this one for you. So let me just do this one quickly for you.

But I will not draw the arrows, just to save time.

So this one, you count how many contracted lines? One, two, three, four, so this gives you n to the 4th for the area. And this I will not do it now. You do it yourself. So, also, if you count the g dependents, this gives you g to the power of 4. And this gives you n squared, g to the power of 4. Is it g to the power of 4? Maybe I'm counting wrong. One, two, three, four. Four propagators is g squared. Yeah. g to the power of 4. Yeah. Yes?

AUDIENCE: So in general, the more-- the more crossings, would that decrease n?

HONG LIU: Yeah, that's right. OK. So now you can just continue. With three vertices, with four vertices, et cetera. But of course, the smart fellows will not stop, and they ask, can I find some rules in doing this?

So, how do you get the general n counting? And whether that's the old planar diagram with the same, or does one need to classify them further? OK. So these are the obvious questions. So let me just tell you some observations. Then we are closed for today. Then we will close for today.

So the first observation.

So just from those examples, let me just draw some observations. Observation one-- so this one requires a little bit of ingenuity. It says, even a and b-- so this a and b-- they cannot be drawn on the plane. They cannot be drawn on the plane without crossing lines. But a and b can be drawn on a torus without crossing lines.

So now let me show it to you. So what is a torus? So we can draw a torus on the plane, like follows, so we draw a square. Then we identify this side with that side. Identify this side with that side. Then this gives us a torus. OK. Because when you do that, that give you cylinder. And then you do that again, and it gives you a torus. But on the plane, just simple identification.

So let's try to see this diagram. Now let me show this diagram can be drawn on the torus without crossing, because the vertex is like this. We want to contract with this guy with that guy, and that guy with that guy, right? Not labeling lines. But now I can just go up. Now this is identified with here, then I can come back here. Can go here, come back here. So this is contracted. And I don't have to cross lines. Yes? Good.

Similarly, let me do that one. Yeah, that one, maybe I will let you do yourself. Yeah, do the-- so this a-- no, this is not a and b. This is b and d. Let me call this-- b, this is-- let me call it c and d. So a and c are planar diagrams and b and d are non-planar diagrams. OK.

So this is b. So try to do it yourself, the d. So you essentially you have two vertex. How to make that non-planar diagram to be planar on this torus when you identify here with there, and there with there? OK. So this observation one. Actually, let me just do it for you.

AUDIENCE: [LAUGHTER]

HONG LIU: So we connect because I need to do-- because it's better I have this example to make my second point. So now I need to connect this to that guy, but I can do it by going up. I can-- this guy can go here. And then go there. Similar thing, OK. And then this is-- I can connect them without crossing lines. So this is observation one.

So now, observation two, again, is a little bit ingenious, but maybe slightly easier if you have done it long enough.

For all the a to d, the Fourier rule is true. The power of n is equal to number of faces. So this one, we call-- so even though they cannot be crossed, they cross nice on the plane, but we can actually straighten them out on the torus. OK. So here we can straighten them out on the torus.

So the second [INAUDIBLE] is a power of n, it's the number of faces in each diagram. After we straighten it out.

OK. First, let's look at this diagram. This is a planar diagram. So there's one face. There's two faces. And then there's a face outside, because there's a whole plane. So there's three faces. So this is n cubed. And the b, on the torus, there's only one face, because each region is connected to each other. OK. So there's only one face. You see the power n. So now you see you had one face, you have two faces, we have three faces, we have four faces, so this is n to the power of 4. And for this one, you have one face corresponding to here. And then you easily convince yourself everything outside this region, they're connected through the identification, so you get n squared. OK. Then n squared.

So there's two observations. So let me stop here. And then, from here, you can figure out the general rules for everything.

[APPLAUSE]