

# 8.821/8.871 Holographic duality

MIT OpenCourseWare Lecture Notes

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## Lecture 4

### Reminder from last lecture

In general geometry, we can identify temperature in this way: we first analytically continue time to imaginary time  $t \rightarrow -i\tau$ , and then from regularity argument find the period of the imaginary time  $\beta$ , which can be interpreted as the inverse of temperature.

In the Schwarzschild black hole case

$$ds^2 = -f dt^2 + \frac{1}{f} dr^2 + r^2 d\Omega^2$$

After transforming to the imaginary time, the near horizon geometry becomes  $\mathcal{R}^2 \times S^2$  (Fig. 1). In order for the metric to be regular at the horizon,  $\tau$  has to be periodic

$$\tau \sim \tau + \frac{2\pi}{K}$$

which gives the temperature of the black hole:

$$T = \frac{\hbar K}{2\pi} = \frac{\hbar}{8\pi G_N m}$$

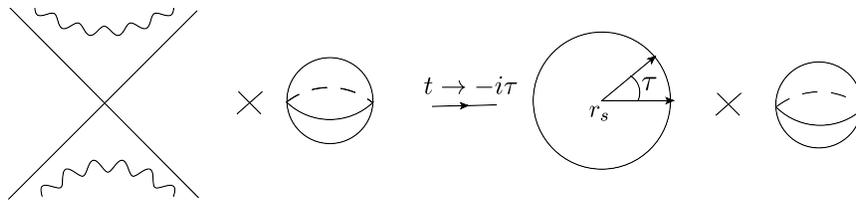


Figure 1: Schwarzschild black hole near horizon geometry in Minkowski (left) and Euclidean (right) signature.

Another example is the 2d Rindler coordinates

$$ds^2 = -\rho^2 d\eta^2 + d\rho^2$$

After making the transformation  $\eta \rightarrow -i\theta$ , the space becomes 2d Euclidean space in polar coordinates (Fig. 2). To avoid a conical singularity at the origin, we must have

$$\theta \sim \theta + 2\pi$$

which gives the local temperature

$$T_{loc}^{Rindler}(\rho) = \frac{\hbar}{2\pi\rho} = \frac{\hbar a}{2\pi}$$

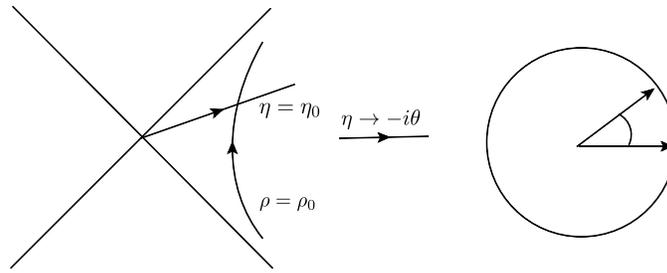


Figure 2: Rindler spacetime in Minkowski (left) and Euclidean (right) signature.

### Physical interpretation of the temperature

Consider a QFT in a black hole spacetime. The “vacuum” state obtained via this analytic continuation procedure from the Euclidean signature is a thermal equilibrium state with the stated temperature.

Remarks:

1. The choice of vacuum for a QFT in a curved spacetime is not unique. The procedure we described corresponds to a particular choice. In the Schwarzschild black hole case, it is the “Hartle-Hawking vacuum”; while in the Rindler case, it is the Minkowski vacuum reduced to the Rindler patch (reduced density matrix of the Minkowski vacuum).
2. If for a black hole, in Euclidean signature we take  $\tau$  to be uncompact, then it is the Schwarzschild vacuum (Boulware vacuum). This is the vacuum that one would get by doing canonical quantization in terms of the Schwarzschild time  $t$ . In the Rindler case, if we take  $\theta$  to be uncompact, we have Rindler vacuum, which can be obtained by doing canonical quantization in Rindler patch in terms of  $\eta$ .
3. In the Schwarzschild vacuum, since the corresponding Euclidean manifold is singular at the horizon, physical observables are often singular there, *e.g.* stress tensor blows up there. But in Lorentz signature, this is not the case. For the “Hartle-Hawking” vacuum, for which all physical observables are warranted to be regular at the horizon. Similar remarks apply to the Rindler and Minkowski vacuum for Rindler spacetime.

### Physical origin of the temperature

We will now use the example of Rindler spacetime to:

1. show that  $\theta \sim \theta + 2\pi$  corresponds to the choice of Minkowski vacuum.
2. illuminate the physical origin of the derived temperature, similarly for the black hole case.

In other words, we will show: the vacuum in the Minkowski spacetime appears to be a thermal state with the temperature:

$$T = \frac{\hbar a}{2\pi}$$

to a Rindler observer of a constant acceleration  $a$ .

Preparations:

1. We now give two descriptions of a thermal state, using a harmonic oscillator as an example.
  - In a thermal state, the thermal expectation of a physical observable can be evaluated as

$$\langle X \rangle_T = \frac{1}{Z} \text{Tr}(X e^{-\beta H}) = \text{Tr}(X \rho_T)$$

where the thermal density matrix reads

$$\rho_T = \frac{1}{Z} \sum_n e^{-\beta E_n} |n\rangle \langle n|$$

here  $|n\rangle$  represents the  $n$ -th excited state, and the partition function is  $Z = \sum_n e^{-\beta E_n}$ .

- In 1960's, H. Umezawa constructed a quantum field theory at finite temperature. Let us follow his idea: we consider two copies of the same system:

$$\mathcal{H}_1 \otimes \mathcal{H}_2$$

where  $\mathcal{H}_1$  and  $\mathcal{H}_2$  correspond to Hilbert space of  $H_1$  and  $H_2$ . Hence typical states in such a system can be written as

$$\sum_{m,n} a_{mn} |m\rangle_1 \otimes |n\rangle_2$$

If we consider a special entangled state:

$$|\Psi\rangle = \frac{1}{\sqrt{Z}} \sum_n e^{-\frac{\beta}{2} E_n} |n\rangle_1 \otimes |n\rangle_2$$

Then the reduced density matrix for subsystem  $\mathcal{H}_1$  is

$$\text{Tr}_2(|\Psi\rangle\langle\Psi|) = \frac{1}{Z} \sum_n e^{-\beta T} |n\rangle_{11} \langle n|$$

This exactly the thermal density matrix  $\rho_T$  in system 1, and any physical observable  $X(1)$  which acts only on  $\mathcal{H}_1$  has the expectation value

$$\langle\Psi|X(1)|\Psi\rangle = \frac{1}{Z} \sum_n e^{-\beta E_n} \langle n|X|n\rangle = \langle X\rangle_T$$

Here the temperature arises due to ignorance of system 2.

We have some additional remarks:

- This framework applies to any quantum systems
- The state  $|\Psi\rangle$  is invariant under  $H_1 - H_2$ , *i.e.*  $e^{it(H_1 - H_2)}|\Psi\rangle = |\Psi\rangle$ .
- We can also express  $|\Psi\rangle$  as

$$|\Psi\rangle = \frac{1}{\sqrt{Z}} e^{-\frac{\omega\beta}{2} a_1^\dagger a_2^\dagger} |0\rangle_1 \otimes |0\rangle_2$$

where  $a_1$  and  $a_2$  correspond to the annihilation operator in  $H_1$  and  $H_2$ .

- One can show that

$$b_1|\Phi\rangle = b_2|\Psi\rangle = 0$$

where

$$\begin{aligned} b_1 &= \cosh \theta a_1 - \sinh \theta a_2^\dagger & \cosh \theta &= \frac{1}{\sqrt{1 - e^{-\beta\omega}}} \\ b_2 &= \cosh \theta a_2 - \sinh \theta a_1^\dagger & \sinh \theta &= \frac{e^{-\frac{1}{2}\beta\omega}}{\sqrt{1 - e^{-\beta\omega}}} \end{aligned}$$

The above transformation is known as the Bogoliubov transformation. So we have  $|\Psi\rangle$  as the “vacuum” for oscillators  $b_1, b_2$ ; just like  $|0\rangle_1 \otimes |0\rangle_2$  is the “vacuum” for  $a_1, a_2$ .

## 2. The Schrodinger representation of QFTs

Consider a scalar field  $\phi(\vec{x})$ , the Hilbert space is all possible field configurations  $\mathcal{H} = \{\Psi[\phi(\vec{x})]\}$ , and the transition amplitude from field configuration  $\phi_1$  at time  $t_1$  to the field configuration  $\phi_2$  at  $t_2$  can be written as

$$\langle\phi_2(\vec{x}), t_2|\phi_1(\vec{x}, t_1)\rangle = \int_{\phi(t_1, \vec{x})=\phi_1(\vec{x})}^{\phi(t_2, \vec{x})=\phi_2(\vec{x})} D\phi(\vec{x}, t) e^{iS[\phi]}$$

And the vacuum wave functional can be obtained by

$$\langle\phi(\vec{x})|0\rangle = \Psi_0[\phi(\vec{x})] = \int_{t_E < 0}^{\phi(t_E=0, \vec{x})=\phi(\vec{x})} D\phi(\vec{x}, t) e^{-S_E[\phi]}$$

Now we come back to a QFT, say a scalar theory, in Rindler spacetime:

$$ds^2 = -dT^2 + dX^2 = -\rho^2 d\eta^2 + d\rho^2$$

Going to Euclidean signature:  $T \rightarrow -iT_E, \eta \rightarrow -i\theta$

$$ds_E^2 = dT_E^2 + dX^2 = \rho^2 d\theta^2 + d\rho^2$$

With  $\theta \sim \theta + 2\pi$ , Euclidean analytical continuation of Minkowski and Rindler spacetime coincide. So Euclidean observables are identical in the two theories. On the other hand, when analytically continued back to Lorentzian signature, for Minkowski spacetime, Euclidean correlation functions becomes correlation functions in the Minkowski vacuum; For Rindler spacetime, we will only have correlation functions in the Minkowski vacuum for operators restricted to the Rindler patch. *i.e.*

$$\mathcal{H}_{Rindler} = \{ \Psi [\phi_R(X)] | \phi_R = \phi(X > 0, T) \}$$

here we have Rindler Hamiltonian  $H_R$  with respect to  $\eta$ , and  $|n\rangle_R$  denotes a complete set of eigenstates for  $H_R$  with  $E_n$ ,  $|0\rangle_R$  is the Rindler vacuum.

For the Minkowski spacetime

$$\mathcal{H}_{Mink} = \{ \Psi [\phi(X)] | \phi = (\phi_L(X), \phi_R(X)) \} = \mathcal{H}_L^{Rindler} \otimes \mathcal{H}_R^{Rindler}$$

here we have the Minkowski Hamiltonian  $H_M$  with respect to  $T$ . Here we denotes the Minkowski vacuum as  $|0\rangle_M$ .

The Minkowski vacuum wave functional

$$\Psi_0 [\phi(X)] = \Psi_0 [\phi_L(X), \phi_R(X)] = \int_{LHP}^{\phi(T_E=0, X)=\phi(X)} D\phi(X, T) e^{-S_E[\phi]} = \int_{\phi(\theta=-\pi, \rho)=\phi_L(X)}^{\phi(\theta=0, \rho)=\phi_R(X)} D\phi(\theta, \rho) e^{-S_E[\phi]}$$

The above expression can also be reduced in the Rindler notation (Fig. 3)

$$\Psi_0 [\phi(X)] = \langle \phi_R | e^{-i(-i\pi)H_R} | \phi_L \rangle = \sum_n e^{-\pi E_n} \chi_n [\phi_R] \chi_n^* [\phi_L]$$

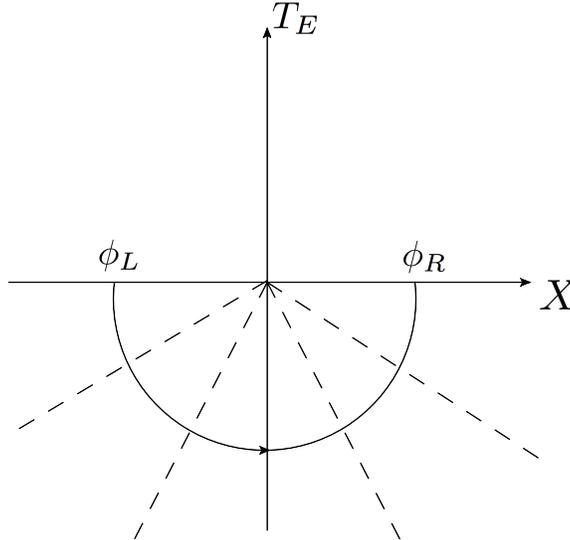


Figure 3: Minkowski vacuum wave functional expressed in the Euclidean Rindler coordinates.

where  $\chi_n [\phi] = \langle \phi | n \rangle$ . Here  $\chi_n^* [\phi_L]$  can also be thought as  $\tilde{\chi}_n [\phi_L] \in \mathcal{H}_{\widetilde{Rind}}$ , and  $\mathcal{H}_{\widetilde{Rind}}$  is Hilbert space with respect to an opposite time direction to the original Rindler space we start with (Fig. 4). Thus we have

$$|0\rangle_M \propto \sum_n e^{-\pi E_n} |n\rangle_{Rind} \otimes |n\rangle_{\widetilde{Rind}}$$

Then trace over the opposite time direction Rindler space, we have the reduced density matrix for the normal Rindler space

$$\text{Tr}_{\widetilde{Rind}}(|0\rangle_{MM}\langle 0|) = \rho_{Rind}$$

And this density matrix itself can also be viewed as a thermal density matrix  $\rho_{Rind}^T = \frac{1}{Z_{Rind}} e^{-2\pi H_R}$  with the inverse temperature  $\beta = 2\pi$ .

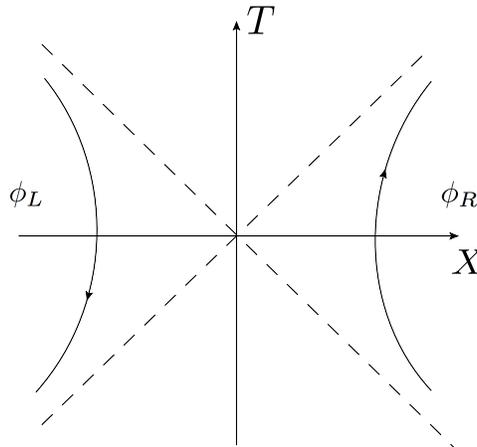


Figure 4: Minkowski Hilbert space as the direct product of left Rindler Hilbert space and right Rindler Hilbert space (but with opposite time direction).

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