Chapter 3: Duality Toolbox

MIT OpenCourseWare Lecture Notes

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Lecture 21

Let us summarize some important results from last lecture. Consider a bulk scalar field $\Phi(x, z)$ with mass m. In $z \to 0$ limit, the behavior of Φ is

$$\Phi(x, z) \to A(x) z^{d-\Delta} + B(x) z^{\Delta} \tag{1}$$

where

$$\Delta = \frac{d}{2} + \nu \qquad \nu = \sqrt{\frac{d^2}{4} + m^2 R^2} \tag{2}$$

The correspondence between boundary CFT operator $\mathcal{O}$ and bulk field Φ works as

$$\text{scaling dimension} = \Delta \tag{3}$$

$$\text{source for } \mathcal{O}\text{: } \phi(x) = A(x) \tag{4}$$

$$\langle \mathcal{O}(x) \rangle = 2\nu B(x) \tag{5}$$

In the example we consider $B(k) \propto A(k)$, *i.e.* $\langle \mathcal{O}(x) \rangle = 0$ if $\phi = 0$. In the presence of source $\phi(k)$, the general result for 1-point function is

$$\langle \mathcal{O}(k) \rangle_\phi \sim \phi + \phi^2 + \cdots \tag{6}$$

In particular, at linear level,

$$\langle \mathcal{O}(k) \rangle_\phi = G_E(k) \phi(k) \tag{7}$$

where

$$G_E(x) = \langle \mathcal{O}(x) \mathcal{O}(0) \rangle \implies G_E(k) \tag{8}$$

by Fourier transformation is the 2-point function, which can also be computed as

$$G_E(k) = \frac{\delta^2 S}{\delta\phi(k)\delta\phi(-k)} = \frac{\delta}{\delta\phi(k)} \langle \mathcal{O}(k) \rangle_\phi = \frac{\langle \mathcal{O}(k) \rangle_\phi}{\phi(k)} = \frac{2\nu B(k)}{A(k)} \tag{9}$$

All above can be generated to other types of operators and corresponding fields.

For higher point functions, recall

$$\log Z_{CFT}[\phi] = S_E[\Phi_c|_{\partial AdS} = \phi] \tag{10}$$

We can consider, for instance, the action as

$$S = -\int d^{d+1}x \sqrt{g} \left(\frac{1}{2} (\partial\Phi)^2 + \frac{1}{2} m^2 \Phi^2 + \frac{\lambda}{3} \Phi^3 \right) \tag{11}$$

where $\lambda \sim \kappa \sim O(1/N)$ ($G_N \sim \kappa^2$). Now we need to solve a nonlinear equation of motion to get classical solution

$$\Box\Phi - m^2\Phi - \lambda\Phi^2 = 0 \tag{12}$$

with

$$\lim_{z \to 0} z^{\Delta - d} \Phi(x, z) = \phi(x) \tag{13}$$

Since λ is small, one can solve (12) perturbatively in $\phi(x)$ and get

$$\Phi_c = \Phi_1 + \Phi_2 + \cdots \tag{14}$$

where Φ_1 is linear in ϕ and Φ_2 is quadratic in ϕ. Substitute this solution back to the action, we must get

$$S[\Phi_c] = S_2[\phi] + S_3[\phi] + \cdots \tag{15}$$

where S_2 is quadratic in ϕ and S_3 is cubic in ϕ, which contain 2-point function and 3-point function respectively. In practice, of course it is tedious to go through this. But this is almost the same as standard perturbation theory in a QFT: we use Feynman diagrams!

Recall in a flat space QFT, how we calculate correlation functions. Consider the $\lambda\Phi^3$ theory in flat space as (11). To get

$$\langle\Phi(x_1)\Phi(x_2)\cdots\Phi(x_n)\rangle \tag{16}$$

Using $\exp(W[J]) = \int D\Phi \exp(S_E + \int d^{d+1}x J(x)\Phi(x))$, and

$$\langle\Phi(x_1)\Phi(x_2)\cdots\Phi(x_n)\rangle = \frac{\delta W}{\delta J(x_1)\cdots\delta J(x_n)} \tag{17}$$

it is equivalent to calculate the following Feynman diagram:

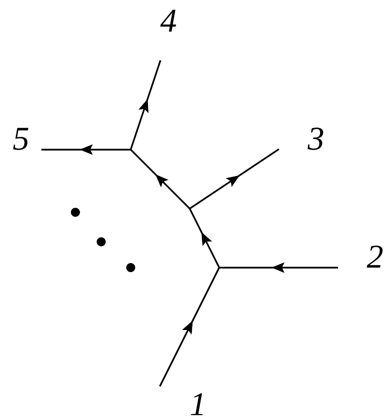

Figure 1: Feynman Diagram in flat space

Now back to AdS, one major difference is source $\phi(x)$ lies on the boundary. Then our Feynman diagram should be as follows:

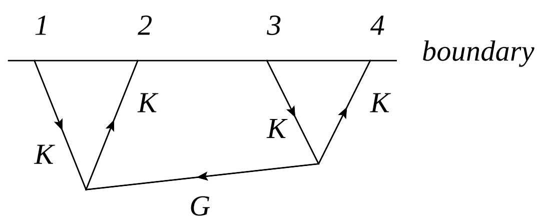

Figure 2: Feynman Diagram in AdS

In the picture, the bulk-to-bulk propagator $G(z, x; z', x')$ is given by

$$(\Box - m^2)G(z, x; z', x') = \frac{1}{\sqrt{g}}\delta(z - z')\delta^{(d)}(x - x') \tag{18}$$

which is the counterpart of standard flat space propagator. In particular, $G(z, x; z', x')$ should be normalizable when either of z or z' is taken to the boundary, *i.e.* $G(z, x; z', x') \propto z^{\Delta}$ as $z \to 0$. This is the result of propagator construction from the quantization of normalizable modes. Furthermore, we must also introduce boundary-to-bulk propagator $K(z, x, ; x')$, which satisfies

$$(\Box - m^2)K(z, x; x') = 0 \tag{19}$$

$$K(z, x; x') \to z^{d-\Delta}\delta^{(d)}(x - x') \qquad (z \to 0) \tag{20}$$

$$\Phi(z, x) = \int d^d x' K(z, x; x')\phi(x') \tag{21}$$

such that Φ computed above behaves like $z^{d-\Delta}\phi(x)$ near the boundary. The analogue of K in flat space is LSZ formula when dealing with external legs.

To summarize, the n-point function in CFT can be calculated as

$$\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle = \langle \Phi(x_1)\cdots\Phi(x_n)\rangle \tag{22}$$

where the right hand side can be computed by Feynman diagrams in AdS with end points lying on the boundary.

Remarks:

1. The full partition function can be separated as classical part and quantum fluctuation:

$$Z_{CFT} = \int_{\Phi|_{\partial AdS}=\phi} D\Phi e^{S_E[\Phi]} = e^{S_E[\Phi_c]}\int D\phi e^{S_E[\Phi_c+\phi]-S_E[\Phi_c]} \tag{23}$$

where $S_E[\Phi_c]$ corresponds to tree-level diagrams and ϕ integral is loop diagrams that can be captured by standard Feynman rules.

2. The complete analogue of standard flat space Green functions are $\langle\Phi(z_1,x_1)\cdots\Phi(z_n,x_n)\rangle$ that only includes bulk-to-bulk propagators. It is natural to expect

$$\langle\Phi(x_1)\cdots\Phi(x_n)\rangle \propto \lim_{z_1\to 0}\cdots\lim_{z_n\to 0}\langle\Phi(z_1,x_1)\cdots\Phi(z_n,x_n)\rangle \tag{24}$$

This boils down to finding the relation between $K(z,x;x')$ and $\lim_{z'\to 0} G(z,x;z'x')$, which we will discuss more explicitly in pset. The crucial result is

$$\langle \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle = \lim_{z_1\to 0} 2\nu_1 z_1^{-\Delta_1}\cdots\lim_{z_n\to 0} 2\nu_n z_n^{-\Delta_n}\langle\Phi(z_1,x_1)\cdots\Phi(z_n,x_n)\rangle \tag{25}$$

3.1.7: WILSON LOOPS

Wilson loops

$$W[C] = Tr\mathcal{P}\exp[i\int_C A_\mu dx^\mu] \tag{26}$$

are most non-local operators in a gauge theory. Here C is a closed path in space time, $A_\mu \equiv A_\mu^a T^a$ where T^a is often in fundamental representation and $\mathcal{P}$ is path ordering. The physical meaning of Wilson loops is phase factor associated with transporting an "external" (quark) particle in a given representation along C. The simplest observable related to it is $\langle 0|W[C]|0\rangle$, although we can also consider for some vacuum with temperature $\langle\Psi|W[C_1]W[C_2]\cdots|\Psi\rangle$. An often used loop is as follows

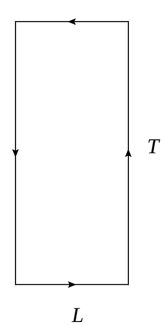

Figure 3: Square Wilson loop

In this picture $T \gg L$. From Wilson loop calculation in QFT, we know $\langle W(C)\rangle \simeq e^{-iET}$ where E is the potential energy between an external "quark" and "anti-quark".

How to calculate $\langle W(C)\rangle$ in $\mathcal{N}=4$ SYM using gravity? First we need to understand how to introduce an external quark in $\mathcal{N}=4$ SYM and its AdS description. Suppose we have $N+1$ D3 branes piled upon each other. If we separate one of them along one perpendicular direction for distance $|\vec{r}|$ (shown in the following picture), the open string connecting those D3 branes will break symmetry from $SU(N+1)$ to $SU(N)\times U(1)$ and we will have some strings with two end points located on the separated D3 brane and the rest N ones respectively. If we consider the fluctuation field living on those D3 branes, this gives a description of a particle ("quark") in fundamental representation of $SU(N)$ with mass $M = \frac{|\vec{r}|}{2\pi\alpha'}$ from symmetry breaking.

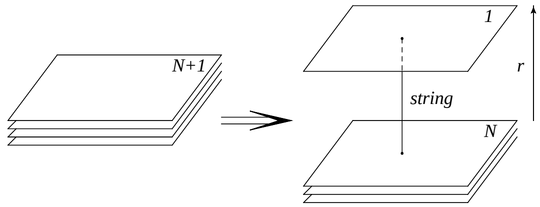

Figure 4: D3 brane separation

Now consider the low energy limit of Maldacena, $\alpha' \to 0$ and $r \to 0$ keeping r/α' finite such that remaining in $\mathcal{N} = 4$ SYM. In the resulted gravity side, N D3 branes have disappeared, one finds only one D3 brane in $AdS_5 \times S^5$ which located at $\vec{r}$ and the other N D3 branes disappeared at $r = 0$ such that we get a "string" hanging from the D3 brane at $\vec{r}$ to $r = 0$. If we want the "quark" to have infinite mass, we should take $r \to \infty$, *i.e.* to the boundary of AdS. In this case, the external "quark" with $M \to \infty$ in $\mathcal{N} = 4$ SYM can be interpreted in bulk as a "string" hanging from the AdS boundary to deep interior and the hanging point is the location of the quark.

8.821 / 8.871 String Theory and Holographic Duality
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