PROFESSOR: So last time we talked about how to describe say for the boundary CFT is at a finite temperature what would be its gravity description? So what we described was a thermal theory. Say a Thermal CFT on the flat space, OK? We show that this is related to a black hole with a planar horizon. The key is that the horizon should have the same topology as the boundary space. So in terms of the picture this is your boundary, so this is z equal to 0, and then the horizon somewhere say, at some value of z0. So horizon has some value of z0 and then the horizon have the same topology as the boundary on space.

And then you can do some of the [? lamicks, ?] et cetera. Any questions about that? Good. So previously we also briefly discussed that the Ads also allow you to put the field theory on the sphere. So if you put the field theory on the sphere then that will be due to the global Ads. Now you can ask the same question. What happens if we put the boundary theory on the sphere and then put there the finite temperature. So let us put a CFT on the sphere at the finite temperature.

So the story here is much richer for a very simple reason. For a very simple reason. So if you have a CFT just on the Rd minus 1 so this Rd minus 1, this euclidean space, euclidean flat space does not have a scale and the CFT does not have a scale. So if you put it at a finite temperature then the temperature is the only scale, and essentially provide a unit over the scale. So the temperature is the only scale, so essentially providing an energy unit. So this means that for CFT on the finite temperature, CFT, on the flat space then physics at all temperatures are the same.

So they are just related by a scaling of units. So in some sense you can see it seems that the physics is very simple whether you have like 0.0001 degree or you have 10,000 degrees, doesn't matter. And it's only the relative scale that matters. So the physics at all temperatures is the same. There's no difference between the low temperature and the high temperature, and the temperature just provide the units.

But if you put the CFT on the sphere now the story's different. So for CFT on the sphere because the sphere has a side itself. OK, so let's take the sphere have both sides r. Again when you put the CFT on the sphere the sides of the sphere does not matter because again the sides of the sphere are essentially provide by the units because the theory's getting [? warrant. ?] So let's just put all the sides to be r. So let's take sides to be r. And the r can be chosen to be the same as the curvature radius in the [? back ?] just to make formula simple, and you can choose it to be any radius.

And now since the sphere already have [? all ?] sides and now you put it at a finite temperature. Now you have the dimensionless number. Then at the finite temperature in the physics is controlled by a dimensionless number, r times t. So now the temperature makes a difference. Now different temperature that r is is different because now you have a relative scale and if rt is small then essentially you're at the low temperature, and then when rt is big then you have a high temperature. Now there's a dimensionless number to characterize whether you're in the low temperature or in the high temperature.

So the physics essentially become much richer because of the and then you have a you have a whole parameter the physics can depend on. And so indeed the story become rather intricate whether we can see the finite temperature theory on the sphere so now let me just mention some important features. So I will walk you not too slowly but also not too quickly about the physics on the sphere. We cannot afford to do it very slowly at this time. Yeah, but if I do it too quickly, of course you won't learn anything.

OK, so let me just point out some important features. So first, when we talked about the story last week-- when we talked about the finite temperature on the flat space-- we said, oh, probably there are two possible descriptions for finite temperature theory from the [? back ?] point of view. So one is that you can have a thermal gas. You have a thermal gas in Ads. But then we argue that for a theory on the sphere that's not allowed because you encompass singularity because the metric becomes singular.

And the one big difference for the theory on the sphere is that actually now this is allowed. So the thermal gas in Ads is now allowed. So now let's go through the argument. So now first let me just write down the global Ads metric. Of course you can write in many different coordinates. So the most convenient coordinates to write global Ads for our purpose is the following. So this is just a pure Ads and r goes to 0 to infinite. So this is like somewhat analog of a, say, sprinkle coordinate in flat space. But the difference is of course now you have nontrivial factors which tell you're in the curve space time.

So now to go to the thermal Ads we do the trick you normally do for thermal field theory. Is that you go to euclidean space and then you put the euclidean time to be periodic in the inverse temperature. Actually, let me write down here the analogous metric on the sphere. So this is a situation we can see that before for the theory on the flat space, on the plane, and then that would be the metric to be on the sphere. So we've talked about last time if you make tau to be periodic then as z goes to infinity then this circle shrink to zero sides. Because the prefactor goes to 0. So the tau circle will have a sides beta but the proper densities controlled by the prefactor. And because when z goes to infinity then the proper lens of the circle goes to 0. So whenever you have a circle we shrink to 0 then you have a singular behavior, et cetera, and then the metric's singular. So you can have [? where ?] it is a singular behavior and so that's why this is not the amount.

But now in this case, when you do such energy configuration this becomes well defined. Because of the proper sides of the tau circle so now you local proper lens local proper sides over the tau circle, it's just given by this prefactor 1 plus r squared divided by 1 squared by r squared beta. And since a greater equal here so this is greater equal to beta. So actually this is bounded from below. So this is a circle [? label ?] will become too small. Essentially the minimal side of the circle is the same as the beta itself. So this euclidean metric is perfectly well defined. OK, so euclidean metric well defined.

So you can trust here that the euclidean metric is singular as z goes to infinity. If you put the 0 in the flat space then that's not allowed but in this case allowed. So for the serial in the sphere-- but still there's an analog of a black hole solution. You can also find the black hole. Again you just write down the standards, sprinkle's metric answers, for black hole metric and then you solve Einstein equation, then you find-- so let me just write down the answer. So you find that this space also allows a black hole solution. Can be written as the following.

So now f just slight generalization of this factor. So the mu is primarily related to black hole mass. And the horizon is that i equal to r0 and r satisfy that this factor goes to 0. So you can also find out what is temperature associated with this black hole. So here that beta can be anything. There's no constraint on the beta. So here using the standard technique to find what is the Hawking temperature for this black hole. Let me just write down the answer. So this is just 4 pi divided by f prime. You've already add to the horizon standard formula and you just find the 0 of this guy, and then you just take the derivative so you find the beta can be written in terms of r0 as follows. So this is just a simple algebra.

So you find that if you take the derivative, we express the derivative is r0, then you find the Hawking temperature of the following form. I'm not doing this calculation here. You can use it to check yourself. Good, any questions so far? So now it seems like we have a problem. Because we have a black hole we also have a standard thermal Ads. Then which one is the right description for the field theory at the finite temperature? Seems like we have a choice.

But now if you look at this function the story's even more tricky than that. So this product of this function this is a rough behavior of this function. So this is the inverse temperature and then this is the sides of the horizon. And the side of the horizon essentially determines your entropy. So it's really a physical thing here and because of the area over the horizon it's the entropy. The area of the horizon is a entropy. So this r0 it can be considered as something standard for entropy.

So now just to get the intrusion, this is a complicated function, it's because intrusion and how this beta depend on this r0, the horizon side. Suppose horizon side is very small, r0 goes to 0, then downstairs is finite. Upstairs goes to 0 so you just go to 0. And the r0 is very large so the downstairs proportionate to r0 square but upstairs only proportionate to r0 so when r0 is very large we go to 0 as the r0.

And this is a small function so there must be a maximum somewhere. And because you can easily find the maximum just to take the derivative of that function. Anyway, so that's how the beta will depend on r. But now this plot is highly peculiar. Why? Can you tell me why?

There's a maximum on theta, which means? That's right. So there's a beta max. So that means this is a beta max which tells you there's a minimum temperature. That means that black hole solution only exist above a certain minimal temperature. And you can easily find out what is that minimal temperature just by taking the derivative of this function and it require to be 0.

Is there any other thing peculiar regarding this plot?

AUDIENCE: When r0--

PROFESSOR: Hmm?

AUDIENCE: When r0, [INAUDIBLE] black hole.

PROFESSOR: Well, r0 is 0-- the black hole becomes smaller and smaller. It means there's no black hole.

AUDIENCE: But the temperature is infinity.

PROFESSOR: Hmm?

AUDIENCE: But the temperature there is infinity. That's weird.

PROFESSOR: No, that's not weird. No, that's the standard, the flat space black hole behaves that way. Just look at a Schwarzschild black hole in flat space when the horizon side becomes smaller and smaller the temperature become higher and higher. That's a standard feature of flat space.

AUDIENCE: It must be the other way. If you have a huge black hole, then why is it that the beta goes down? That seems bizarre. Maybe it can't also be standard, right?

[LAUGHTER]

PROFESSOR: Both things you said are correct and are important but there's something more elementary you're not pointing out. You're pointing out the higher order of important things. But there's one lower order important thing that which is not yet pointed out.

AUDIENCE: There are two solutions of the same temperature.

PROFESSOR: Exactly. In terms of the temperature this is a [INAUDIBLE] So if you look at the given temperature they're actually two solutions. So for any given temperature there are two solutions. Say T greater than T min so this is for the beta smaller than beta max means that T greater than T min There are two solutions. And then we have to label them so let's very imaginatively call this one the big black hole and then this one the small black hole. Because this one have a larger size and this one has a smaller size. So this corresponding to the r0 of those two.

So you have two black hole solutions. One have a larger radius than the other one. So we have big black hole and a small black hole. And now this is an important thing you mentioned that for the small black hole the temperature when you decrease the sides the beta decrease. That then means the temperature increases. So this as I said it's like an entropy. So if you lower the entropy somehow your temperature increases.

So when you increase the temperature you actually lower the entropy for the small black hole. A small black hole you go like this. Go like this. When you reduce the sides, you reduce the beta, means you increase the temperature. But this is actually the standard Schwarzschild black hole behavior in flat space. And this just tells you have [? a lack ?] of specific heat.

So this is a situation we understand. This is a situation we can come up with a flat space, a Schwarzchild black hole. And it make sense why it's happening here because here roughly is order r. Because the r is the scale here so the maximum location where the maximum is controlled by the r. So this region corresponding to sides of the black hole to be much, much smaller than Ads curvature radius.

So when something is much smaller than the curvature radius what you do see? Hmm?

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah, flat space. Yeah, it just like even though we live in the curved universe but we actually see flat space because of course our scale is much more than curvature radius of universe. And a similar key thing here even though the black hole have a size much smaller than the curvature radius of Ads they essentially behave like flat space black hole. And so that's exactly what you see for the flat space black hole.

And indeed, in a more interesting is this big black hole, the [? interesting ?] is a big black hole. And the big black hole when r0 increases the temperature increase. So this is the right way. So this is indeed the standard, just positive specific heat. You can check it has positive specific heat. So this is indeed what you'd expect from a thermal system. And so this is indeed what you expect from a thermal system.

So now we are encountering an even weirder situation. We not only have so more ideas. We're not only have black hole. We actually have two black holes. So what's actually described with the field theory at finite temperature? So what does this really mean? Number two is a black hole solution. Now let's look at the number three.

So now let's look at what are the indications those solutions. So first thing you can just guess. So what do you think what these three solutions should mean? If you believe this, duality has to be true. Yes, so that the game we play here is that you believe that the duality is always true. And then whenever you see some phenomenon, the gravity side, you say this, I try to integrate. I try to understand on the field theory side. And it makes sense and then we just carry out the guess and check it, et cetera.

So what would the three solution mean? Three gravity solutions, or in principle for example, for a temperature at here we can have three solutions all have the same temperature. What would this mean? So the simplest guess is that it tells you that for CFT and the ST minus y at this temperature there's three possible phases. There's three possible phases. It's just like water, ice, or vapor. There's three possible phases.

Now, the interpretation is clear. Then we should just find. So to decide what is the right solution then we should find what? That's right, we should find the phase, find the one with the lowest free energy. And then to decide which is the right one then we just need to find the one with the lowest free energy. So now recall that the partition function the field theory side should always be identified with the partition function on the gravity side. And then this side the partition function by definition is minus beta, the free energy. [? Seems ?] like going more and more. that's weird. But on the gravity side we write this by saddle-point approximation. We integrate over all possible field on the gravity side weighted by the euclidean action, and the leading order in the saddle-point approximation you just evaluate it. You just evaluate it on the classical solutions.

So now you can just equate them. You just need the free energy must be equal to se to the to the to the [? Ic. ?] So this is just the same principle we discussed before on how we just apply it. So now if we believe the duality and we believe with some of the [? lamicks ?] then we should just find the solution with now just SE.

So we should calculate the euclidean action for all three solutions and then we find the corresponding free energy for each of them, and then the one with the largest, SE, will correspond to the one with lowest free energy. Then that will be the stable solution. And the other phases presumably will be unstable or not stable. just like in the standard story.

So this conclusion was drawn from some of the [? lamrick ?] itself. But what will be nice we can actually draw this conclusion without using thermal dynamics. So we say, so this [INAUDIBLE] was drawn to identify this with the free energy and then we say according to the sum of dynamics, and then we should choose the one with the lowest energy. But will be better if you can actually derive this thing from the gravity side, and you can and for a very simple reason.

So this also follows from the standard rules of saddle-point approximation as follows. Oh, I forgot to write on this board. So when you evaluate this [? possible ?] so when you evaluate your integral you're in the saddle-point approximation, so normally you are instructed to add the contribution of all saddle points. So the fact that you have three different solutions it tells you have three separate saddle points.

And then just from the standard rule of the saddle-point approximation you just need to add all of them together. So you add the expression SE, you evaluate the thermal Ads. And the expression, SE you evaluate it as a big black hole, and then you add the SE to the small black hole, et cetera. OK, you add all of them together.

And now each action is in exponential. And as I said, they are weighted. All this action are weighted by n square. Because the action is always proponent to law of [? Newton ?] and the law of newton is [? proponent ?] to n square. So there's a big parameter in the exponential.

So the saddle is not just SE dominates in the large N limit. You just add three exponentials, which one is the biggest will dominate. Because each one of them is proponent to n square and n goes to infinity. And n goes infinite means it's greater than the other guy by a tiny bit. in the prefactor of n squared they will become predominant.

So we see that the sum of the [? lammicks. ?] Sum of the [? lammicks tells ?] us we should define the smallest free energy and then that statement can be essentially derived if I don't know that statement. I can actually derive that rule, say finding the smallest free energy, just by using the standard saddle-point approximation from the gravity side. So this is a nice consistency check. This a nice consistent check that the duality which we believe to be true should be true. Yes?

AUDIENCE: So the one thing I don't totally understand is that, so say that you give me this Euclidean integral over metrics on the gravity side. So the idea that there's really only three contributions-- because we found three solutions with a particular temperature-- how do we know that there aren't any other solutions? Like, how do we know there aren't any other terms that were not accounted for?

PROFESSOR: Oh yeah, there are three-- so this is a rule of the saddle-point approximation, you find the saddle-point. So you find the solution of the equation of motion with the right boundary conditions and these are the only ones. Yeah, and then of course, each of them include the fluctuations. This is just the standard saddle-point approximation. Any other questions? So now the task just boils down to-- yeah, let me just emphasize here-- yeah, so now, the task just boils down to find what is the lowest free energy. OK?

So now it's 3-- no, it's 4. So now, let's try to decide which one is more stable. So first, let me just emphasize SE is proportional to the 1 Newton-- 1/G Newton. And then it's always proportional to N squared. OK? Because 1/G Newton is proportional to N squared if you translate from the [INAUDIBLE] language.

And now, if you vary the free energy for them-- for this-- here let me call this-- thermal [? gas and ?] AdS, I always just call it TAdS, OK, Thermal AdS. I think, yeah, just-- the shorthand is just thermal AdS. So for the thermal AdS, this is actually 0. This is actually 0. So thermal AdS is the free energy-- I should say this way. SE is actually the 0 times N squared.

And then you can have fluctuations then that can contribute to the order and to the power 0 [? contribution ?]. So the 0 in this-- 0 is simple. It is because, when we go from the [? pure ?] AdS into the thermal AdS, essentially we just changing the global structure. OK? We just make the Euclidean circle so for the pure AdS, when you go to Euclidean space, the tau have infinite size.

And then when you go to the thermal AdS, we just make the tau to be a compact circle. So you have not changed the-- the solution is identical. The solution is identical go to the pure AdS. Just the global structure is different in terms of the periodicity of the Euclidean time circle.

So now, if you vary the classical action-- so if you're reference point is that the pure AdS, the classical action is 0. And then, for the thermal AdS will be 0. OK? Will be the same. It would be the same-- the classical action will be the same as pure AdS, OK?

So [INAUDIBLE] order N squared, but then you can have fluctuations which contribute. And the fluctuations is to the-- yeah, so the fluctuations can be interpreted as the, say, the graviton symbol [? and ?] division from the symbol, graviton gas. OK? And so, say, in the themal AdS sites essentially, have a thermal gas graviton.

And there, free energy is independent of N because they are the fluctuations. And just as usual, whenever you can see that the fluctuations there, they're independent of a coupling constant. Is this clear?

AUDIENCE: Why is the first term 0?

PROFESSOR: Hm?

AUDIENCE: Why is the first term 0?

PROFESSOR: The first term is 0 because this has the same classical solution as the pure AdS.

AUDIENCE: The pure AdS, then, has a nonzero [INAUDIBLE].

PROFESSOR: Yeah. Yeah, that's a very good question.

AUDIENCE: [INAUDIBLE]

PROFESSOR: Yeah. I will-- yeah. Wait a few minutes. I will talk about this more precisely. It's a good question.

AUDIENCE: Oh, OK.

PROFESSOR: Yeah. Let me just tell you the answer first, and then I will say a few things. Any other questions? OK. And then if you calculated the Euclidean action for the big black hole and the small black hole, you find that they are always greater than that. It's always larger than action for the small black hole.

And this is some number, nonzero number, times order N squared. So this is always the case. We also [INAUDIBLE] way for you to calculate in a few minutes. But I will not do this explicit calculation here because it takes some time. So there is a simple way, physical way, to understand this-- to understand why this is true because this is a free energy.

So this tells that the free energy of the big black hole is always much more-- is always smaller than the free energy of the small black hole. This is not a precise way to understand it, but it is a heuristic way-- is that if you look at these two solutions-- so these two solutions have the same temperature, but the big black hole have a larger size.

When you have a larger size and the entropy is proportional to the area of the horizon-- and the entropy is much higher, so the big black hole has much higher entropy than the small black hole. And so you choose-- you would expect that this should have a smaller free energy. But, of course, you have to do a precise calculation to really check it because their energy is not the same. Their energy is also not the same. Yeah, but this is a heuristic way to understand.

Anyway, so you find that the free energy-- so you find the Euclidean action, or the [INAUDIBLE], is always much-- is always greater than the small black hole. And that is good. That is good because that tells you the small back hole is never dominate because you always find something bigger than that. And the small black hole have a [? lack of ?] specific heat.

That's not something you want because our Field Theory definitely have quality of specific heat. Our Field Theory definitely has quality of specific heat. So that tells you that this is never-- can be low-- that would be a dominate phase.

So now, if you calculated a number-- so now if you look at the number explicitly--

So what you'd find actually-- so you also have-- let me just first tell you the results, OK? You find there exists a temperature, Tc, which is greater than the minimal temperature. Then, you find that the-- when you're smaller than the minimal temperature--

Yeah. Right. OK. So when your T is smaller than the minimal temperature-- oh, no. No. T is smaller than Tc. So you find that there exists a Tc, which is greater than the minimal temperature. So let me just say this. So when T is smaller than T minimum, of course, there's only thermal AdS exists. And, of course, that would be the only phase which can describe the Field Theory result.

But, now, there exists another temperature [? then, ?] within T, smaller than-- greater than T minimum and smaller than Tc. Then, you find that the SE big black hole is smaller than 0. And the thermal AdS, essentially, by definition to be 0. So that means that the thermal AdS will have lower free energy. So in this range, the thermal AdS will dominate.

And when T is greater than Tc, then you find that the big black hole now has a positive action and [INAUDIBLE] dominates. So if I draw this-- and then there's a Tc and there's a T minimum-- so this is the temperature axis. So here, of course, you only have a thermal AdS. But within this range, the thermal AdS dominates over big black hole and the small black hole.

And when you're above, then the big black hole dominates. So you actually have a transition at Tc. So below here, it's just thermal AdS. And then you have a transition from thermal AdS to the big black hole at Tc.

So a quick side remark. How we-- how would you find this result? So this would be the result if you calculated the-- so you just calculate the Euclidean action. You compare them. That's what you would find.

So any questions about that? Yes?

AUDIENCE: Do you ever get the small black hole phase at all somehow artificially?

PROFESSOR: Yeah, you can always get artificially. Yeah, artificially, always. Yeah. And let me not go into that. So now, let me just say a few remarks, say, in finding SE. So when you find the Euclidean action, you find it is always divergent. It's the same thing as what we encountered before. If you calculate the whole action, you have to integrate the-- over the-- essentially, the volume of the AdS.

But the volume of AdS goes to infinity at-- near the boundary, essentially, you always get infinity. So in order to get a [? finite ?] answer, you need renormalization.

So when you do renormalization, essentially, you put the [? cut ?] off and then you subtract covariant, the local counter terms. You put a cutoff. And then you subtract covariant counter terms at the cutoff. And then you get a finite answer. Then, you take the cutoff to the boundary.

So we will not go through this procedure, but it's something you can do. But there's an alternative shortcut to not go through these. Or you can just subtract-- just calculate the Euclidean action for the cutoff, then subtract the value of pure AdS.

So if you calculate the pure AdS, you will find the answer is also divergent. You will find the answer is also divergent. And you just subtract it. And then when you do the subtract, you will find the difference is [? finite. ?] And then you find the difference is finite.

And this is a slightly simple way to do than that, but we also not do this because, still, you have to calculate Euclidean action [? and such. ?] So the synchron-- shortcut is to assume thermal dynamics. That's what we did last time. Because you can find the entropy density very easy. So entropy density-- did I erase my-- so the entropy density is just, essentially, the area of the horizon.

So the entropy-- right now, it's actually the total entropy, not the entropy density because now we are [? almost here. ?] So the entropy would be-- and size of the sphere will be r0 to the power d minus 1 times omega d minus 1. Omega is the volume of a unit sphere. And then divided by 4 pi 4GN. So this is the entropy of the system.

And you can express this in terms of the temperature because r0 is a function of temperature. r0 is a function of temperature. If you invert this, you can imagine r0 is a function of temperature. So you have-- essentially you have entropy as a function of temperature. And-- Hm?

AUDIENCE: d minus 1, what's that?

PROFESSOR: [? Formula ?] d minus 1 is the volume of the unit sphere.

So you know the entropy. You can just use the formula S equal to minus F minus T divided by T to-- so you can also write this as minus F r0, partial r0, partial T, et cetera. And then you can integrate this equation to find the-- to find F as a function of r0 because we know how the r0 depends on T from here. So you just need to integrate this equation.

So this is a simple exercise because now you just need to do an integral of some function. And then you find omega d minus 1 divided by 16 pi GN, r0 d minus 2 minus r0 d divided by R squared. You get actually a rather simple answer if you do that integration.

So we have chosen the integration constant so that when r0 equals to 0, this is 0 because r0 equal to 0, if the black hole has 0 size, essentially, [? there's ?] no black hole. And then we have chosen the free energy to be 0 for the [? no ?] black hole.

So this free energy, again, should be interpreted as the difference with the pure AdS. So for the black hole-- this is the black hole free energy because, by definition, this way, the pure AdS, you just have 0.

Any questions about this? So now you see here, clearly, this expression. So now this is a free energy now. This is not the Euclidean action. So they differ by the amount [? assigned ?] in my convention. So now you see that there exists r0, critical r0. F black hole is greater than 0 Free energy is greater than 0 for r0 is smaller than R.

So you can write it as formula d minus 1-- so let me just write. Yeah, you can put the omega-- the r0 to that [? out ?] for the overall factor r0. Yeah, put d minus 2 out, and this just becomes 1 minus r0 squared.

So you find that the free energy is greater than 0. But r0 is greater than 1. That's greater than the-- so this is greater than the value of the thermal AdS. And the black hole is smaller than 0. The free energy [? smaller ?] than 0 become great than R.

So the r0 is somewhere here. So r0 is somewhere here. It has to exist above the-- it's somewhere here. This is r0. This is this r0 critical [? as you go ?] to R. This is r0 critical [? as you go ?] to R, somewhere here. So it's always a big black hole.

And you can find out what is this critical temperature. So the critical temperature is just the beta evaluate at r0 equal to R. And then you can find out that this is 2 pi R divided by d minus 1. Just using this formula you can find out it's that.

So now I emphasize that this is a phase transition. And this is not only a phase transition, this is actually a first order phase transition for the following reason. So in our unit, below here, the free energy is 0 times order N squared. It's 0 times N squared. But about here-- well, actually, above Tc-- at Tc, the free energy is exactly equal to 0.

So [? r0c ?] is equal to R at Tc. There's a free energy. It's exactly 0. So at Tc, the-- a big black hole and the thermal AdS have the same free energy. So they can exist together. And above here, then the free energy of the big black hole is some negative number times order-- times N squared.

So your free energy has a huge change. So essentially, the first derivative of free energy, is discontinuous across the phase transition. So this is actually a first order phase transition. So this is-- yeah, let me just emphasize that. So this is a first order phase transition, with the free energy equal to order N to the power 0 when T is smaller than Tc and some nonzero number times N squared. Well, T greater than Tc.

Any questions?

AUDIENCE: When r0 is equal to R, then F is equal to 0.

PROFESSOR: Yeah.

AUDIENCE: Then, why is the-- oh.

PROFESSOR: No, the derivative of F should be discontinuous. No, F is continuous. No, this is a definition of the-- for the phase transition, the free energy is always continuous. And then the free-- so the first derivative here will be proportional to N squared. And you can check-- Yeah?

AUDIENCE: So the point is-- OK, so just see if I understand. So the point is that it's discontinuous in the limit of large N or something? I mean--

PROFESSOR: Yeah, that's right. That's right. Exactly. So let me emphasize. So this is N goes to infinity limit. Is there any other questions? So we see there's a lot of [? rich ?] story when you go to the sphere. And, actually, there's a phase transition. And the phase transition roughly [? adds to-- ?] the black hole size is the AdS radius. Exactly the black hole size is AdS radius.

And this is, more or less, what you expected because that's where the physics become nontrivial because when the horizon is much, much smaller than the black hole [? size-- ?] than the curvature radius, as I said, just should reduce to the flat space black hole and et cetera. Yeah, because the curvature radius is only scale here.

So now, let me make another remark. Now, I erase this. I think it's OK. Yeah So you should actually check yourself. So below Tc, the F prime is always 0. When you take the derivative, it's always 0 at order N squared. Let me, again, write this answer, 0 times N squared. So you should check that the first derivative of F is discontinuous.

AUDIENCE: So F itself is not discontinuous?

PROFESSOR: No, F is not because the F is 0 precisely at T equal to Tc. So F is equal to 0. And the-- Yeah. So the reason the first derivative is discontinuous is very clear because below Tc, this is 0. So the first derivative is 0 times N squared. And above Tc, the derivative is the derivative of the free energy of the black hole at I equal to r0. And, clearly, the derivative is nonzero.

At I equal to r0, the derivative is [? nonzero. ?] OK? And so you see, they're discontinuous.

So since physics only depends on the [? dimensionless ?] number, R times T-- So large R, small T-- large R at fixed T essentially is the same as the, say, large T fixed R. It can only depend on the ratio of them. OK? It depends on the product of them. So you can either think of-- you can either think you fix the temperature, you increase the size of the sphere. Or you fix the size of the sphere, you increase the temperature.

[? In ?] fact, the same is to increase this guy. And this is essentially the limit going to the flat space. It's going to the R to the T. If you take R go to infinity, you go to a [? theory ?] on R d minus 1. So we see that the theory on R d minus 1 is mapped to the high temperature limits of the theory on the sphere.

So essentially, just corresponding to the 1 point for the [? theory ?] on the sphere and this infinite temperature of the [? theory ?] on the sphere. So that's why, as we said before, because there's no scale here, the [INAUDIBLE] temperature from here is essentially the same in the flat space. And in particular, this is described by a big black hole.

And this is exactly what we see before because when the black hole become very big, then you can approximate it to a plane because, locally, it's like a plane [? anymore. ?] And the spherical horizon is just like a plane. And then you go to the black plane-- you go to the black hole with the flat horizon.

AUDIENCE: Why are the phases only dependent on R times T?

PROFESSOR: Yeah, because this a CFT and this is only dimensions number. It can only depend on them through this dimensions number. There is no other scale. Good? Other questions? Yes?

AUDIENCE: So on the side of the CFT, what are the different phases? I didn't-

PROFESSOR: Sorry?

AUDIENCE: What are the different phases on the CFT side?

PROFESSOR: Yeah, that's what we are going to explain. So now, we have to describe the black hole story. Or now, we have described the gravity story. Just by looking at the gravity size, we saw there are three possible phases. At low temperature, you get this thermal AdS phase. At high temperature, you get big black hole phase, and then there is a phase transition between them. And then, also, you have unstable small black hole phase, which never appears as a stable phase at any temperature.

So, now, let's try to see whether we can understand this from the Field Theory side. What does this mean from the Field Theory side? Why somehow-- if I, for example, put [INAUDIBLE] on a sphere, do I expect such kind of phase transition? Does this make sense? Oh, by the way, I should say this is the-- so this is called a Hawking-Page transition. So this is called a Hawking-Page transition.

So remarkably, this was discovered in 1980-- I think '81 or '82, almost 20 years before this AdS/CFT conjecture. But they already figured out that there is a phase transition. They were-- so they were looking at the black holes [? and ?] AdS, and they said, oh, there's two black holes. But there's also some AdS. And then, somehow, there's some kind of phase transition between them. But because they don't know the Field Theory, they don't know this should correspond into some kind of Field Theory system. But they figured out this gravity story essentially in 1981.

Now, let's explain what should be the Field Theory interpretation of this-- The Field Theory explanation of this. Physical reasons for Hawking-Page transition.

So now, I will consider a toy example. So I won't consider-- so [INAUDIBLE] on the sphere, that's a little bit too complicated. But I'm going to consider toy example. And you will see, from this toy example, that the physics behind this is very simple and actually, also, [INAUDIBLE].

So let's consider you have N squared harmonic oscillator-- free harmonic oscillator. So let's just imagine you can take them to be different frequencies-- [? let's ?] just even for simplicity-- or for the same frequency. Say, omega-- I have some omega, which I just take to be 1. Doesn't matter.

So just consider N squared free harmonic oscillator. So I claim when N-- when N goes infinite limit, this system have the same phase transition-- have exactly the same phase transition described there. So, now, I will explain. So first, let's look just at-- this is a system we know how to solve exactly, so let's look at the spectrum.

So this is total spectrum, the total energy spectrum of this N squared harmonic oscillators. So let's call the 0, the ground state. So let me normalize the energy so that the ground state have energy 0. And then you can excite one harmonic oscillator, and then you have state of 1. Yeah. Actually-- Yeah, have state of 1.

And you can have two harmonic , oscillator et cetera. And then you'll have many harmonic oscillators, say, of all-- then, you're going to have almost every harmonic oscillator excited. And then you have energy of order N squared, et cetera.

So now you have to imagine a slightly nontrivial condition. Now, you have to imagine a slightly nontrivial condition, which this is the only thing [? reach ?] beyond the harmonic oscillator. Yeah. Actually, I just realized this.

I slightly oversimplified the story a little bit. Yeah. Yeah, let me say this way. Yeah. Yeah. Let's consider 2N harmonic oscillators. So let me arrange it into two matrix, A and B. And they are all free. it doesn't matter. For this purpose, it doesn't matter how many harmonic-- just imagine I have arranged them into two matrices, so I have two N squared harmonic oscillators.

And then I can excite them. But, now, I have a condition. So I have a condition, which is analog to the condition of the gauge invariance. Instead, I want all the states to be the trace singlet created by A and B. You have A, B, et cetera. So A have N squared creation and annihilation operator, and B have N squared annihilation-- creation and annihilation operator.

And then you can act them-- then, you can [? accurately ?] form the spectrum that it is. But there's a-- but also I need to add a gauge invariance condition. It's that the state [? acted ?] by A and B have to be SU(N) singlet. And SU(N) is the conjugate. Just imagine A and b transformed under some SU(N) [? join representation. ?] And they have to be the SU(N) singlet.

So equivalent statement say that the trace A and B-- the order state has to be created by operators inside the trace, single trace or multiple traces, et cetera. So is this clear?

AUDIENCE: So one thing. So A and B are operators? Or they are--

PROFESSOR: They're operators, harmonic oscillators.

AUDIENCE: Oh, I see.

PROFESSOR: Yeah. Imagine you have two matrices under-- and two harmonic oscillators that form two matrices. And each of them have N squared harmonic oscillator. And if I just say they are free, then you don't even have to think about the matrix structure.

AUDIENCE: So the only thing, I'm confused about the matrices. So are the matrices acting on some [INAUDIBLE] So in other words--

PROFESSOR: Yeah. Let me be a little bit more explicit. [INAUDIBLE] Think about the foreign system. The Lagrangian 1/2 Tr A dot squared, Tr 1/2, Tr B dot squared minus 1/2 Tr A square 1/2 Tr B squared. So this is a free harmonic oscillator. So these are two N squared free harmonic oscillators with frequency 1.

So if I don't impose this singlet condition, this is just a purely free harmonic oscillator. But now, I [? require ?] all the state-- and so SU(N) transformation-- [? yeah, you can act them. ?] But now, I impose the condition that they will have-- they should be SU(N) singlets. So they can [? create ?] it by A and B, et cetera. But they should be SU(N) singlet. Good?

So now one thing you can convince yourself is that because of the trace condition-- so if you think about the state of energy 1 or 2, et cetera-- so as far as the energy is order N to the power 0, so as far as the energy does not scale with N, you can check yourself the density of state-- the way you can-- the degeneracy of those states will always be of order N to the power 0. So this is a fact. You can convince yourself.

And when you go to the energy of order N squared, then you can check that the density of state become exponential of order N squared. So the reason for this is very simple. So the reason for this is very simple heuristically. So the reason for this is very simple because if you have a state of order 1, which does not scale with N, means there's only order 1 states-- order 1 oscillators are excited.

And then, of course, the density of state will be independent of N. And if you do have energy of order N squared because each of-- each oscillator has a frequency of 1, and then you have energy N squared. And then that means order N squared oscillator is excited. And then you have order Ns-- if you have order N squared oscillator excited, then how many ways you can choose them to construct the [? state of ?] energy order N square and that [? exponential ?] of order N squared.

So imagine each of the states, you can have N possible oscillator to act, say, for each of them for twice. Then, you have 2 to the order N squared probabilities of doing that. So that's where this [? exponential ?] order N squared come out. So this is a fact. You should try to convince yourself if it's not obvious to you now.

So the reason I add this singlet condition is for here. If I don't add a singlet condition, then there's some kind of independence here, which I don't want. There can be [? log ?] independence. Yeah, just-- yeah. So this makes [? your ?] story a little bit [? convenient. ?] Oh, so now-- yeah. We're running out of time unfortunately.

So now let's consider the free energy, which, roughly, you can consider as integrates of all possible states with this weight factor and times the density of state. [INAUDIBLE] because the free energy sum of all states, minus BH, and the sum translated into the integral with a density of states. And then that translates into that.

So naively-- so we always consider-- so our beta is always-- does not scale with N. That's order 1. It's always order 1. So if you look at here naively, only state of energy of order that's not scale N will contribute significantly because of this suppression.

Because of the thermal suppression, say, if you have energy of state of order N squared, then this is a huge suppression. Then, they should not contribute. Then, they should not make a visible expression-- a contribution to your free energy. Expect, except when those states have a huge density of states. So except when those states have a huge density of states.

So, here, we see those states have actually order N squared density of states. So that means, in here, in that integral, you have beta times order N squared-- something of order N squared and then times something positive of order N squared. When you can see the contribution of the state of energy of order N squared.

So when this factor dominate over this factor, then the entropy will overwhelm the thermal suppression. And then, such state will dominate. And that precisely happens when beta becomes small enough. Because when beta becomes small enough, when you got the high temperature, then beta decrease. So eventually, this will overwhelm this one because the beta will decrease. And that's precisely the Hawking-Page transition we'll see.

So at low beta-- at big beta or small temperature, thermal-- this one dominates. And then, the [? V ?] would be of order N to the power 0 because only state of order-- energy order zero will contribute. But the beta-- when beta is sufficiently small, then the [? log ?] D(E) minus BE can become greater than 0. And then the order N squared states will dominate.|

And then you find the-- then, you find the [? partial function ?] will be [? exponential ?] of order N squared. And that's precisely what we see in the Hawking-Page transition. So I emphasize that physics here have nothing to do with the details of the system, only related to the large N and to the density of states. So if you have interactions, if you have more complicated system, it doesn't matter.

It doesn't matter. As far as your [INAUDIBLE], this phenomenon will happen. This phenomenon will happen. And so this is the essence of the Hawking-Page transition. We will stop here.