

Holography Duality (8.821/8.871) Fall 2014

Assignment 3

Oct. 10, 2014

Due Friday, Oct. 24, 2014

- Please remember to put **your name** at the top of your paper.

Note:

- The classical string action and light-cone quantization are discussed in many places, for example,
 - Zwiebach, “A First Course in String Theory,” Chap. 6-13.
Barton’s discussion is very pedagogic and has many nice details, which should be very useful for you to build up intuitions. Chap. 14 is a nice, simple introduction to superstring.
 - Green, Schwarz and Witten, “Superstring theory,” Vol. 1, Sec. 2.1 and 2.3.
 - Polchinski, “String Theory,” Vol. 1, Sec. 1.2 and 1.3
 - Polchinski, “Joes Little Book of String,” available at:
<http://www.kitp.ucsb.edu/sites/default/files/users/joep/JLBS.pdf>
This is Polchinski’s class notes which aimed to provide a watered-down version of his beautiful, although some formal book. The notes also contain a simple introduction to superstring at the end.
- D-branes are discussed in Chap. 15 of Barton’s book mentioned above. Polchinski’s book Chap. 8 introduces D-branes via toroidal compactification and T-duality, which lies outside what we covered in lectures. Polchinski’s recent lecture notes mentioned above also contains a nice discussion of D-branes.
- Chap. 4 of my book with Krishna and other friends, *Gauge/String Duality, Hot QCD and Heavy Ion Collisions* published earlier this year by Cambridge University Press (expanded from a previous review arXiv:1101.0618), was an attempt to convey key conceptual aspects of string theory to people outside string theory without going into any technical details. One potential pitfall of such an attempt is that it might be too trivial for experts, while too concise (or lack of details) for non-experts, thus ending up being useful for nobody. I would greatly appreciate your candid comments on that chapter to get a sense of how successful our experiment is.

Problem Set 3

1. Conserved World-sheet currents (20 points)

Consider the Polyakov action for a string moving in a D -dimensional flat spacetime in the conformal gauge (i.e. the worldsheet metric $\gamma_{ab} = \eta_{ab}$)

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \partial_a X^\mu \partial^a X_\mu \quad (1)$$

The above action has global symmetries corresponding to the Poincare symmetries of the target spacetime. Find the corresponding worldsheet conserved currents associated with these symmetries and give physical interpretation of them.

2. Virasoro algebra (35 points)

In lecture we showed that for open string with Neumann boundary conditions moving in a D -dimensional flat spacetime, the equations of motion for X^μ can be solved to give

$$X^\mu = x^\mu + 2\alpha' p^\mu \tau + \sqrt{2\alpha'} \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-in\tau} \cos n\sigma . \quad (2)$$

We will work with the light-cone gauge $X^+ = 2\alpha' p^+ \tau$. Below for convenience we will also use the notation $\alpha_0^\mu \equiv \sqrt{2\alpha'} p^\mu$.

- (a) Show that in the light-cone gauge, the constraints $T_{ab} = 0$ can be solved explicitly by expressing p^- and α_n^- in terms of modes of transverse coordinates. In particular, introducing *transverse Virasoro operators*

$$L_n = \frac{1}{2} \sum_i \sum_{m=-\infty}^{\infty} : \alpha_{n-m}^i \alpha_m^i : \quad (3)$$

where i sums over all transverse directions and $::$ denotes normal ordering, show that (for all n including $n = 0$)

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha'} p^+} (L_n - a\delta_{n,0}) . \quad (4)$$

a is a normal ordering constant which we argued to be given by $a = \frac{D-2}{24}$ using zeta-function regularization. $n = 0$ equation of (4) gives the mass-shell condition we discussed in detail in lecture.

(b) Using the commutation relations of α_n^i , which are

$$[\alpha_m^i, \alpha_n^j] = m\delta_{m+n,0}\delta_{ij}, \quad (5)$$

show that the Viraroso operators introduced in (3) satisfy the Viraroso algebra

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \quad (6)$$

with

$$c = D - 2. \quad (7)$$

(c) Find the commutators

$$[L_n, X^i(\tau, \sigma)] = \quad (8)$$

and show that L_n generate diffeomorphism on the worldsheet, i.e.

$$\epsilon[L_n, X^i(\sigma, \tau)] = X^i(\sigma^+ + \epsilon\xi_n^+, \sigma^- + \epsilon\xi_n^-) - X^i(\tau, \sigma) \quad (9)$$

where ϵ is a small constant parameter and

$$\sigma^\pm \equiv \tau \pm \sigma, \quad \xi_n^\pm \equiv -ie^{im\sigma^\pm}. \quad (10)$$

(d) Give physical interpretation of what you found in (b) and (c). Could you explain why L_n should generate the specific diffeomorphism of equation (9).

Note: Viraroso algebra is arguably the most important algebra in string theory. In the light-cone gauge, for consistency we should verify that the conserved charges for Lorentz symmetries derived in Prob. 1 should indeed satisfy the Lorentz algebra. The Viraroso algebra plays an important role in this calculation and one finds that the Lorentz algebra is recovered only for $D = 26$.

3. Scalar modes as transverse fluctuations of D-branes

In lecture we mentioned that massless scalar fluctuations of a D-brane can be interpreted as fluctuations of transverse positions of the D-brane. While this may seem intuitively simple, the physical implications are profound and lie at the heart of magic of string theory. So it is perhaps useful to review the logic once more here. We started by imposing Dirichlet boundary conditions on some of the coordinates of an open string. We then worked out the most general classical solutions of a string under such boundary conditions. At this stage (i.e. classically) the boundary conditions are rigid and strings are simply restricted to move on a submanifold in spacetime. The magic comes when we quantize the string and discover that there is a massless scalar field for each Dirichlet direction. Now these scalar fields are interpreted as the collective coordinates for the shape of the submanifold. In particular, their *condensates* correspond to

translating this submanifold around in transverse spacetime directions. Thus the original rigid boundary conditions have now turned into a full dynamical object, which can fluctuate and move around.

This is in fact the same reasoning which leads to the realization that string theory gives rise to quantum gravity. Classically we just describe a closed string propagating in a *rigid* spacetime (for example, a flat Minkowski spacetime). Upon quantization, we discover a massless spin-2 mode which is interpreted as a graviton. Now the spacetime in which string propagates is no longer rigid, but can fluctuate and becomes dynamical; in other words, we have quantum gravity. Also condensates of gravitons can lead us away from the flat spacetime to a general curved spacetime.

A remarkable thing is that string cannot just propagate consistently on any given spacetime; Weyl invariance on the worldsheet should be maintained, which in turn determines the dynamics for the spacetime in which the string propagates. Imposing Weyl invariance for closed string then leads to a generalized Einstein equations which the metric and other possible background fields should satisfy. Even for flat spacetime, this requires $D = 26$ for bosonic string.¹

For the case of an open string with Dirichlet boundary conditions, the Weyl invariance determines the dynamics of the motion of the D-brane. Consider, for example, a D0-brane, i.e. with Neumann condition in the time direction X^0 and Dirichlet conditions in all other directions X^i . One can show that if we impose the boundary condition

$$X^i(\tau, \sigma = 0, \pi) = f^i(X^0(\sigma, \tau)) \quad (11)$$

which, according to our interpretation above, should describe the motion of the D0-brane following a trajectory $x^i = f^i(t)$. We use x^i and t to denote spacetime coordinates to distinguish from X^i and X^0 which denote the corresponding fields on the string worldsheet. The Weyl invariance of the string worldsheet then implies that $f^i(t)$ should satisfy the equation

$$\partial_t \frac{\dot{f}^i}{\sqrt{1 - (\dot{f}^i)^2}} = 0, \quad \dot{f}^i \equiv \partial_t f^i(t) . \quad (12)$$

The above equation precisely corresponds to that of a point particle with an action

$$S = \int dt \sqrt{1 - (\dot{f}^i)^2} . \quad (13)$$

¹In the light-cone gauge we were using, this condition appears from imposing Lorentz invariance. In other covariant gauges in which Lorentz invariance is maintained throughout, then the condition $D = 26$ comes from imposing Weyl invariance on the worldsheet.

I will not ask you to derive (12), although I was very tempted to. If you are interested, you are welcome to give a try. You can also consult the paper: R. Leigh, Mod. Phys. Lett. A4, 2767 (1989) for a derivation of a generalization of (12).

4. Separation of D-branes as Higgs mechanism (30 points)

Consider two D24-branes sitting at $X^{25} = 0$ in $D = 26$ dimensions. There are four different types of open strings labelled by their end points.

- (a) Consider separating one of the branes to $X^{25} = a$. Write down the mode expansion for an open string stretched between the two branes.
- (b) Work out the excitation spectrum for an open string stretched between the two branes.
- (c) Now focus on the $N = 1$ modes for all types of open strings, where N is the total excitation number. These are the massless modes when the branes are coinciding and can be described by a gauge field A_α and a scalar field ϕ in the adjoint representation of $U(2)$. When the branes are separated some of these modes become massive as you have found in (b). Show that the spectrum of these modes (i.e. mass and degeneracies) is consistent with that from the Lagrangian

$$\mathcal{L} = \text{Tr} \left(-\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{2} (D_\mu \phi)(D^\mu \phi) \right) \quad (14)$$

in a Higgs phase with expectation value

$$\phi = \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix}, \quad v \equiv \frac{a}{2\pi\alpha'}. \quad (15)$$

In (14) the Tr is over $U(2)$ matrices.

Note: Lagrangian (14) is defined in the worldvolume of the D24-branes, i.e. $(24 + 1)$ d. It can be obtained from the pure Yang-Mills Lagrangian in $D = 26$ by taking $A_{25} = \phi$ and letting all fields to be independent of x^{25} . This procedure is called dimensional reduction. The reason for this is intuitively clear; we have discussed earlier that for D25-branes (i.e. Neumann boundary conditions in all directions), based on general principle, the low energy theory for the massless vector modes should be given by a Yang-Mills theory. When changing some directions to Dirichlet boundary conditions, the only difference is that we can no longer have momenta in those directions.

For N Dp-branes, the corresponding Lagrangian can be obtained by doing dimensional reduction of a pure Yang-Mills theory with gauge group $U(N)$ on $25 - p$ transverse dimensions.

- (d) Now consider N D p -branes. Write down the expectation value for the scalar fields in this theory corresponding to these D-branes sitting respectively at

$$\vec{X}^{(k)} = (a_1^{(k)}, \dots, a_{25-p}^{(k)}), \quad k = 1, 2, \dots, N \quad (16)$$

where the superscript k labels different branes and $\vec{X}^{(k)}$ denotes the transverse position for each brane.

5. Mass of a D-brane (15 points)

The mass of a D-brane can be obtained by summing over vacuum diagrams of open strings. (In the closed string case summing over all the vacuum diagrams should give the cosmological constant.)

- (a) Show that this implies that the mass of a D-brane is proportional to $\frac{1}{g_s}$, where g_s is the string coupling.
- (b) Show that in the limit of weak string coupling, the gravitational interaction among D-branes is negligible compared to its mass. Thus we can treat parallel D-brane configuration as almost stable.
- (c) Now consider a stack of N D-branes. Show that its gravitational effect can no longer be ignored when N becomes sufficiently large. Give the order of N in terms of g_s when this happens.

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