

Holography Duality (8.821/8.871) Fall 2014

Assignment 4

Oct. 31, 2014

Due Friday, Nov. 14, 2014

- Please remember to put **your name** at the top of your paper.

Note:

- Nice reviews of conformal field theories include
 - “Applied Conformal Field Theory” by Ginsparg, arXiv:hep-th/9108028.
 - “Conformal field theory” by Di Francesco, Mathieu and Senechal (a book).

Both of the above references focus on conformal field theories in (1+1)-dimension. There are some nice discussions at the beginning regarding conformal symmetries in general dimensions.

The big AdS/CFT review

Aharony, Gubser, Maldacena, Ooguri and Oz, “Large N Field Theories, String Theory and Gravity,” arXiv:hep-th/9905111.

also contains a concise discussion of conformal field theories for general dimensions in sec. 2.1.

- For more on AdS spacetime, see sec. 5.2 Hawking and Ellis, “The large scale structure of spacetime.” The big review by Aharony et al mentioned above also contains a nice review of various useful coordinates in sec. 2.2.
- The discussion of general aspects of AdS/CFT will partially follow Chap. 5 of my own review with Krishna and other friends
“Gauge/String Duality, Hot QCD and Heavy Ion Collisions,” arXiv:1101.0618 (the extended version is now published as a book by Cambridge University Press).
- There are many reviews on AdS/CFT with different focuses. There is no space to list all of them. The big review mentioned above remains a good reference on certain topics (but of course it does not have all the development after 1999). Here are some nice reviews by our own colleagues:
 - E. D’Hoker and D. Z. Freedman “Supersymmetric Gauge Theories and the AdS/CFT Correspondence,” arXiv:hep-th/0201253.

- Chap. 23 of Barton’s book “A first course in string theory” (2nd edition).
- McGreevy, “Holographic duality with a view toward many-body physics,” arXiv:0909.0518.
- Allan Adams, Lincoln D. Carr, Thomas Schaefer, Peter Steinberg, and John E. Thomas, *Strongly Correlated Quantum Fluids: Ultracold Quantum Gases, Quantum Chromodynamic Plasmas, and Holographic Duality*, arXiv:1205.5180.
- D. Z. Freedman and A. Van Proeyen, *Supergravity*, Cambridge University Press. Part VIII contains a nice introduction to AdS/CFT.

Problem Set 4

1. Inversions and special conformal transformations (20 points)

Consider the AdS_{d+1} metric in Poincare coordinates

$$ds^2 = \frac{R^2}{z^2} (\eta_{\mu\nu} dx^\mu dx^\nu + dz^2) \quad (1)$$

- (a) Show that the following transformation (called “inversion”)

$$z' = \frac{z}{\eta_{\mu\nu} x^\mu x^\nu + z^2}, \quad x'^\mu = \frac{x^\mu}{\eta_{\mu\nu} x^\mu x^\nu + z^2} \quad (2)$$

is an isometry of (1).

- (b) Show that the transformation (2) is not continuously connected to the identity.
- (c) Consider the $z \rightarrow 0$ limit of (2) and show that it is a conformal transformation of the Minkowski metric in d -dimension.
- (d) Now consider the isometry (denote it by S) obtained from the following sequence of transformations on $x^M = (x^\mu, z)$: an inversion, followed by a translation along x^μ directions by a constant vector b^μ , and then another inversion. S is an isometry since every step in the sequence is. Write down the transformation S explicitly. (Note: while inversion is not continuously connected to the identity, S is.)
- (e) Taking $z \rightarrow 0$ limit of transformation S , show that this leads to a special conformal transformation of the Minkowski metric.

2. Geodesic motion in AdS (20 points)

Consider AdS in global coordinates

$$\frac{ds^2}{R^2} = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\Omega_{d-1}^2 . \quad (3)$$

Suppose a massive particle starts at $r = 0$ and $t = 0$ with energy E and follow a radially outward geodesic. Find

- (a) the maximal value of r the particle can reach;
- (b) the time t it takes for the particle to come back to $r = 0$.

3. Geodesic distance (20 points)

An AdS_{d+1} space can be represented as a hyperboloid in a flat Lorentzian space-time of signature $(2, d)$:

$$X_{-1}^2 + X_0^2 - \sum_{i=1}^d X_i^2 = R^2 \quad (4)$$

with a metric

$$ds^2 = -dX_{-1}^2 - dX_0^2 + \sum_{i=1}^d dX_i^2 \quad (5)$$

From this construction it is manifest that AdS_{d+1} is homogeneous (i.e. every point is equivalent) and has an isometry group $SO(d, 2)$.

Consider two points A and B in AdS. There are two notions of invariant distances between them. The first is *geodesic distance* $\sigma(A, B)$, which is the distance square defined in terms of embedding space metric (5). The second is the *geodesic distance* $s(A, B)$ defined in terms of a geodesic connecting A and B . Show that (set $R = 1$ below):

- (a) If A, B are connected by a time-like geodesic, then

$$\cos s(A, B) = 1 + \frac{1}{2}\sigma(A, B) . \quad (6)$$

- (b) If A, B are connected by a space-like geodesic, then

$$\cosh s(A, B) = 1 + \frac{1}{2}\sigma(A, B) . \quad (7)$$

- (c) In (6) and (7), $\sigma(A, B)$ lies in the range $-4 \leq \sigma(A, B) < \infty$. Describe what happens when

$$\sigma(A, B) < -4 . \quad (8)$$

4. **IR/UV connection in global AdS (20 points)**

Discuss IR/UV connection for the bulk theory in global AdS. In particular, elucidate the difference with that in the Poincare patch.

5. **Holographic principle: bound on the number of degrees of freedom (20 points)**

Since S_5 is compact, IIB string theory in $\text{AdS}_5 \times S_5$ is essentially a five dimensional gravity theory with 5-d Newton constant given by

$$G_N^{(5)} = \frac{G_N^{(10)}}{V_5} \quad (9)$$

where V_5 is the volume of S_5 and the 10-d Newton constant $G_N^{(10)}$ is related to string coupling g_s and α' by

$$16\pi G_N^{(10)} \equiv 2\kappa^2 = (2\pi)^7 g_s^2 \alpha'^4 . \quad (10)$$

The duality between $\mathcal{N} = 4$ SYM and IIB string in $\text{AdS}_5 \times S_5$ can thus be considered as a realization of the holographic principle in a gravity theory in AdS_5 .

The holographic principle also implies that the total number of degrees of freedom \mathbb{N} inside some region of area A should be given by

$$\mathbb{N} = \frac{A}{4G_N} \quad (11)$$

where G_N is the Newton constant, which in our case should be $G_N^{(5)}$. We would like to check whether (11) is true for the duality between $\mathcal{N} = 4$ SYM and IIB string in $\text{AdS}_5 \times S_5$. From the duality the degrees of freedom in the bulk gravity theory should be given by those of SYM theory. Thus it boils down to check whether the number of degrees of freedom of SYM satisfies (11) with A the area of boundary of AdS.

Here one immediately encounters a difficulty as both sides of (11) are infinite; the SYM theory has an infinite number of degrees of freedom and the area of the boundary of AdS space is also infinite (we will use metric (1)). Thus we need to introduce some cutoffs to make these quantities finite. We take each spatial direction parallel to the boundary to have a finite size of length L and the boundary of AdS to be at $z = \delta$ instead of $z = 0$. Now the area A of the boundary of AdS is finite, given by

$$A = \frac{R^3 L^3}{\delta^3} . \quad (12)$$

Now let us try to estimate the number of degrees of freedom on the SYM side. First, from IR/UV connection, the cutoff $z = \delta$ in the bulk radial coordinate should translate into a short-distance cutoff $\epsilon \sim \delta$ on the SYM side. (Make yourself comfortable of this.) We can then imagine that the SYM theory lives on a lattice of size ϵ and at each site there are cN^2 degrees of freedom, where N^2 comes from $SU(N)$ and c is an $O(1)$ constant.

- (a) Show that (11) indeed works up to an $O(1)$ factor.¹
- (b) Repeat the exercise for global AdS, which is dual to SYM on S^3 .
- (c) Consider in global AdS (as described e.g. by (3)) the spatial region inside some value r_c of radial coordinate r . Show that

$$\lim_{r_c \rightarrow \infty} \frac{V}{AR} = \text{finite} \tag{13}$$

where V and A are the volume and area of the region.

Thus it may seem that in global AdS that (11) works up to an $O(1)$ constant is much less impressive as the volume and area are not that different in AdS. However, I would like to still argue that the agreement for (11) is impressive. Why? State your own opinion on this.

¹There are various reasons why we cannot get this $O(1)$ factor precisely. Firstly the relation $\epsilon \sim \delta$ is only heuristic. Secondly, we are not able to obtain c precisely as the formula (11) is semi-classical in nature, which from mapping of parameters implies that one should work in the strongly coupled regime on the SYM side.

MIT OpenCourseWare
<http://ocw.mit.edu>

8.821 / 8.871 String Theory and Holographic Duality
Spring 2015

For information about citing these materials or our Terms of Use, visit: <http://ocw.mit.edu/terms>.