8.821 String Theory
Fall 2008

8.821 F2008 Lecture 25: Thermal aspects of $\mathcal{N} = 4$ SYM

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1 The $\mathcal{N} = 4$ plasma at large λ

Previously in lecture 23 we gave evidence that the Black Hole thermodynamics of the AdS Black Brane (non-extremal) solution was dual to the thermal ensemble of $\mathcal{N} = 4$ SYM on $\mathbb{R}^3$ at large N and λ. Thus the gauge theory provides the microstates that are being coarse grained by the Bekenstein-Hawking entropy of the black hole S_{BH}.

A few comments on this observation:

1.1 Hydrodynamics

- Perturbing the equilibrium of the *boundary* theory with a kick will result in thermalization - relaxation back to equilibrium.
- In the *bulk* the response to such a kick is for the energy of the kick to fall into the black hole.

The above two statements are related by the duality. In the long wavelength and small frequency limit both are consistent with the hydrodynamics of a relativistic CFT. Additionally the duality allows one to compute various transport coefficients of the gauge theory at large λ, such as the shear viscosity, R-charge conductivity, etc. See the review [1].

1.2 Thermal Screening

At finite temperature T correlators die off exponentially at large separation $r \gg 1/T$, even if in vacuum there are massless fields which mediate long range interactions. This is because such particles develop a thermal mass from continuously interacting with the thermal bath.

For example the force between two external charges in a gauge theory should behave at large distances as,

$$V_{q\bar{q}}(r) \underset{r \gg 1/T}{\sim} e^{-m_{\rm th} r} \tag{1}$$

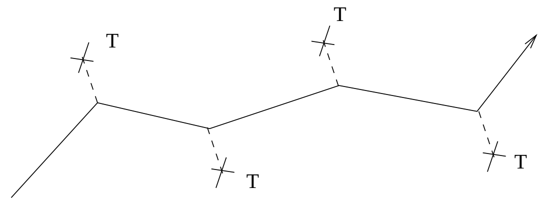

Figure 1: A particle receives multiple kicks from its thermal surroundings. This generates a thermal contribution to the particles mass.

where $m_{\rm th}$ is the thermal mass.

The $q\bar{q}$ potential can be calculated using a Wilson loop,

$$\exp\left(-iV_{q\bar{q}}(r)\tilde{T}\right) = \left\langle P\exp\left(i\oint_{C_r} A\right)\right\rangle_T \equiv \langle W_{C_r}\rangle_T \tag{2}$$

where the contour C_r is a rectangle with spatial separation r and temporal separation $\tilde{T}$ not to be confused with temperature. The expectation value of the Wilson loop is taken in a thermal ensemble. The above formula is defined for $\tilde{T} \gg r$ large.

Of course as we learned in a previous lecture we can compute Wilson loops at large λ using the duality. One must find the minimal action string in the bulk which approaches C_r at the boundary. Then the expectation of the Wilson loop is $\langle W_{C_r}\rangle = \exp(-iS_{\rm min}(C_r))$.

There are two (stable) saddle points to this problem[1] The two string configurations are shown in Fig. 2. For $r < r_*$ the curved string has the smallest action $S_{\rm min} = V_U(r)\tilde{T}$ while for $r > r_*$ the two straight strings win $S_{\rm min} = V_{||}\tilde{T}$ which is independent of r. Where the point of cross over is $r_* \sim 1/T = \pi z_H$.

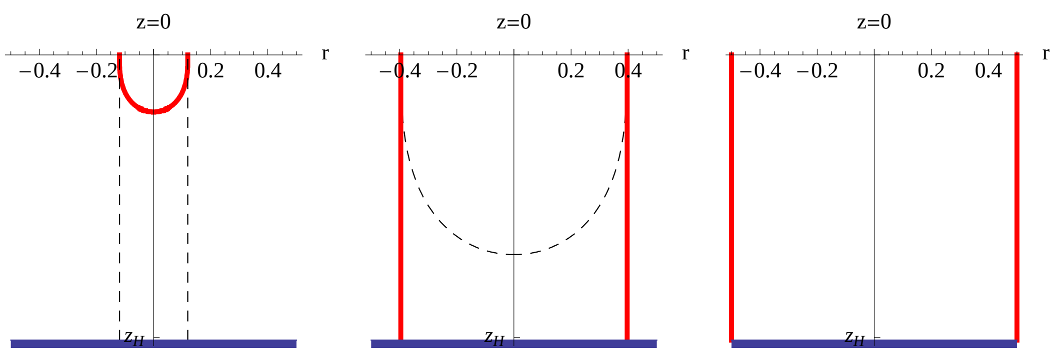

Figure 2: The two string configurations as a function of r (increasing from left to right) with the minimum action configuration highlighted in red. For large enough r the curved string solution ceases to exist (its action becomes complex!)

Both actions diverge at the boundary. To yield a finite result for the $q\bar{q}$ potential we must renormalize by subtracting $V_{||}$.

[1]There is a third (unstable) saddle point, which one can think of as coming from a local maximum of some fiducial potential.

After renormalization the potential looks like,

$$V_{q\bar{q}}^{\text{ren}}(r) = V_{q\bar{q}}(r) - V_{||} = \begin{cases} V_U(r) - V_{||} & r < r_* \\ 0 & r > r_* \end{cases} \tag{3}$$

For small $r \ll 1/T$ the potential approaches the $T = 0$ result; $V_{q\bar{q}}^{\text{ren}} \sim \frac{\sqrt{\lambda}}{r}(1 + c(rT)^4 + \ldots)$. See Fig. 3. Since the potential vanishes for $r > r_*$ there is no trace of the interaction, and the gluons mediating the interaction are screened (more than the expected exponential fall off. [2])

Note that the kink is an artifact of the large-λ approximation.

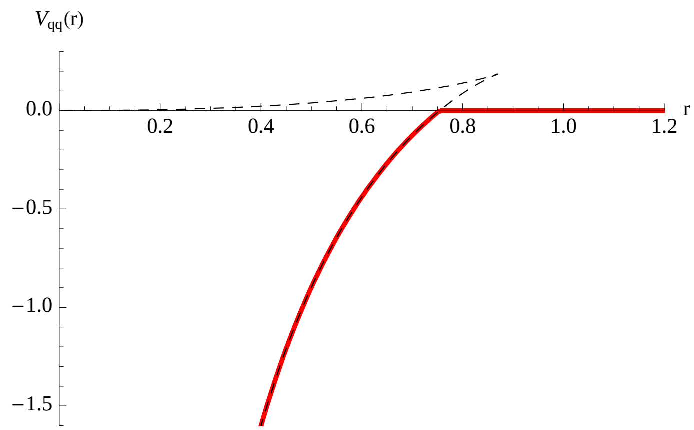

Figure 3: The red line follows the actual potential. The other curves represent string configurations with higher action. In particular we have included a third configuration, the highest curve, which corresponds to the unstable solution mentioned in footnote 1.

1.3 Polyakov-Susskind Loop

The Polyakov loop is an operator defined in the Euclidean theory by,

$$U = P \exp\left(i \oint_{S^1_\tau} A \right) \tag{4}$$

where the contour S^1_τ is around the Euclidean time circle.

The expectation value of this operator gives the free energy in the presence of an external charge in representation R;

$$\langle \text{tr}_R U \rangle = e^{-F_q(T)/T} \tag{5}$$

This gives an order parameter for confinement at finite T,

$$F_q = \begin{cases} \infty & \text{confinement,} \quad \langle \text{tr} U \rangle = 0 \\ \text{finite} & \text{deconfined} \quad \langle \text{tr} U \rangle \neq 0 \end{cases} \tag{6}$$

[2]From this one may conclude that the screening length is r_*. However there is another candidate for the screening length which comes from the next order correction to V^{ren} in the $1/\sqrt{\lambda}$ expansion: the two long strings may still interact via exchange of a massive glueball in the high-T effectively 3+0 confining gauge theory. This results in a potential $\sim e^{-m_{\text{gball}} r}$ where m_{gmall} is the lightest mass glueball of this 3d theory. See [2].

In the gravity dual we can compute $\langle \mathrm{tr}_R U \rangle$ using a string which ends on S^1_τ. The 5 dimensional Euclidean AdSBH has the topology $\mathtt{R}^3 \times$ D where D is a disk with boundary S^2_τ at the boundary of AdS, as in Fig. 4.

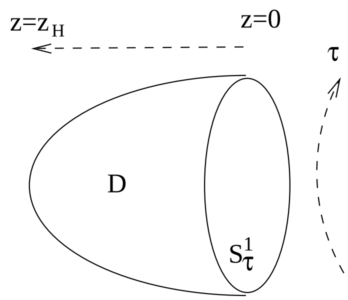

Figure 4: The Euclidean time circle at the boundary S^1_τ closes off smoothly in the bulk Euclidean black hole solution. A string is begging to be placed here.

This geometry follows from the form of the metric,

$$ds^2 = \frac{L^2}{z^2}\left(f(z)d\tau^2 + \ldots\right) \tag{7}$$

Since $f(z)$ vanishes at some $z = z_H$ the Euclidean time circle which is periodically identified shrinks to zero size at this point. [3] So S^1_τ is contractable in the full geometry, and we may wrap a string world sheet on the disk D. It follows that the expectation value of the Polyakov loop is non zero,

$$\langle \mathrm{tr} U \rangle \sim e^{-A_D/\alpha'} \neq 0 \tag{8}$$

where A_D is the area of the disk D. Note actually A_D is infinite, coming from the usual UV divergence associated with an infinite quark mass probe. This can be regulated in the usual way to yield a finite result.

We thus conclude that $\mathcal{N} = 4$ SYM at strong coupling is not confining on $\mathcal{R}^3$ for any temperature T.

2 A large-N deconfinement phase transition.

We now want to consider the more interesting case of $\mathcal{N} = 4$ SYM on S^3 which does indeed exhibit a confinement to deconfinement phase transition. Along the way we will see that the AdS/CFT dual of this phase transition implies a dramatic consequence for Quantum Gravity; we must sum over geometries and topologies (consistent with certain boundary conditions.)

2.1 $\mathcal{N} = 4$ SYM on S^3 - Kinematic Confinement

As opposed to the theory on $\mathtt{R}^3$ this theory has a unique vacuum, $|\Omega>$:

[3] Recall in a previous lecture we used this fact to find the period with which we must identify τ in order for the geometry to be a smooth disk.

- the flat directions are lifted by the conformal coupling term RX^2
- there are no zero modes for the fermions because we have taken antiperiodic boundary conditions for them around the thermal circle (thus breaking SUSY.)
- there are no harmonic 1-forms on S_3 so the gauge fields have no zero energy modes
- A_0 can have a zero mode, however it is not propagating: the equation of motion $\delta S/\delta A_0 = 0$ is the Gauss' law constraint.

Gauss' law on a compact surface implies that all physical states must be color singlets, even under the global part of the gauge group $SU(N)$. This is the (trivial) statement of "kinematic confinement".

We can reduce the theory on spherical harmonics on the S_3 such that the operators in the theory are $M_i \in \{\text{KK modes of:} A_\mu, \Psi, X\}$. All M_i's are in the adjoint representation. So low lying states are given by

$$\text{tr}\,(M_1 \ldots M_S)\,|\Omega > \tag{9}$$

where the trace ensures only gauge singles contribute. The energy of these states is $E = \propto S/R_{S_3}$ and the density of states with fixed S is the number of ways to order and choose the M_i's. This density of states grows exponentially with E, $\rho(E) \sim e^{E/T_{hag}}$ where T_{hag} is called the Hagedorn temperature. For small E, such that $E \ll N^2$, the density of states is independent of N, so at least for small T's only these states are excited and we have,

$$F(T \ll 1/R_{S_3}) \sim \mathcal{O}(N^0) \tag{10}$$

the free energy is independent of N. For higher temperatures states with energy of order N^2 become important in the thermodynamic ensemble (in particular this must happen before the Hagedorn temperature where thermodynamic quantities begin to diverge). The number of such states is now no longer independent of N, simply because there are trace relations for (9) when $S \sim N^2$ and the counting of such states changes.

In order to estimate F for high temperatures we can use the fact that for the CFT of $\mathcal{N} = 4$, the physics only depends on the dimensionless parameter TR_{S_3}. So large T's at fixed R_{S_3} can instead be achieved by taking R_{S_3} large at fixed T. Then the theory limits to a theory on $\mathcal{R}^3$ at fixed temperature. For the free theory we can estimate the free energy as that of a collection of N^2 free fields,

$$F/T \propto N^2 (R_{S_3})^3 T^3 \tag{11}$$

where $(R_{S_3})^3$ is the volume factor. Hence the free energy now scales like N^2, which is a signal of deconfinement. At finite N this deconfinement phase transition becomes smooth, see fig. However, even in QCD ($N = 3$) it is a very sharp, dramatic feature (QCD does not need to be put on a sphere to see this, because QCD already confines on $\mathtt{R}^3$.)

See [3] for more about this.

2.2 Gravity dual- the Hawking Page phase transition

We would like to find the Gravity dual of the situation discussed in the previous section. This was discovered by [5] and applied to the AdS/CFT correspondence by [4].

To find the gravity dual we look for an asymptotically AdS Solution who's boundary is (conformal) to $S^3 \times S'_\tau$. That is look for a local extrema of the action $I[g] = -1/(16\pi G_N) \int d^5x \sqrt{g} \left(R + 12/L^2\right)$.

The answer is the (Euclidean) AdS-Schwarzschild Black Hole, which for convenience later we will call the space X_1:

$$ds^2 = h(r)d\tau^2 + dr^2/h(r) + r^2 d\Omega_p^2 \tag{12}$$

where for $p = 3$,

$$h(r) = 1 + r^2/L^2 - \mu/r^2 \tag{13}$$

For large r this metric becomes AdS in stereo-graphic coordinates (with the boundary at $r = \infty$.)

The mass can be computed using the usual formula,

$$M = \left\langle \int_{S_3} T_t^t \right\rangle \rightarrow \mu = \frac{16 G_N M}{3\pi^2} \tag{14}$$

The horizon is located where $h(r)$ vanishes. There are two roots, we will look at regions outside the largest of the two $r > r_+$.

$$\frac{r_\pm^2}{L^2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{4\mu}{L^2}} \tag{15}$$

The temperature is found by demanding no conical singularity at $r = r_+$:

$$T(r_+) = \frac{2r_+^2 + L^2}{2\pi L^2 r_+} \tag{16}$$

Importantly this $T(r_+)$ has a minimum as a function of $r+$, see Fig. 5: that is there is a minimum temperature $T = T_1 = \sqrt{2}/\pi L$ for which there exist black hole solutions. Also above this temperature there are two types of black holes: large and small.

$$r_+(T)^{\text{large}}_{\text{small}} = \frac{\pi L^2 T}{2} \left(1 \pm \sqrt{1 - \frac{2}{\pi^2 L^2 T^2}}\right) \tag{17}$$

These have the following thermodynamic properties;

- **Small BH**: $r_+^{\text{small}} < L$. For $r_+ \ll L$ they look like black holes locally on $R^{4,1}$, that is Schwarzschild black holes. Their mass satisfies,

  $$M^{\text{small}}(T) \sim r_+^2 \sim 1/T^2 \tag{18}$$

  So the the specific heat is negative $C_V = \frac{dM^{\text{small}}}{dT} < 0$. So not only do Schwarzschild BH's evaporate, this process cannot be hindered by placing them in equilibrium with their surroundings because of this thermodynamic instability: if its surroundings have higher T, the BH eats some energy, grows bigger and decreasing its T.

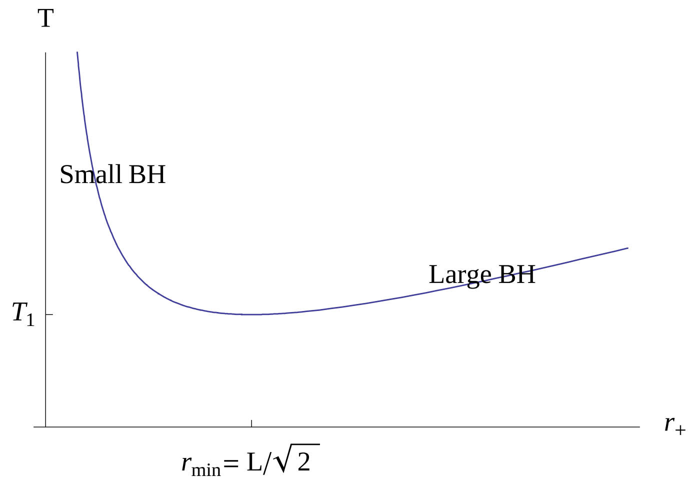

Figure 5: The temperature T as a function of horizon radius r_+.

This is equivalent to Jeans instability in flat space; there is no canonical ensemble for gravity in flat space.

- **Large BH**: $r_+^{\text{large}} > L$ and one finds

$$M^{\text{large}}(T) \sim r_+^4 \sim T^4 \tag{19}$$

such that $C_V > 0$. So a large BH can be in equilibrium with its surroundings, such as its own Hawking radiation.

The heuristic reasoning for this is that AdS acts like a box (an infrared cutoff on the canonical ensemble) for the Schwarzschild black hole. In this case if the surroundings have higher T the Schwarzschild BH will eat some energy, grow larger and decrease its T, however now because the surroundings are finite they will also decrease their T. If this is enough to compensate for the decrease in the black holes T an equilibrium can be obtained.

From now on the space X_2 will refer to the large BH's described here.

For temperatures smaller than $T < T_1$ we have not found a solution with the correct assymptotics. Fortunately there is a solution for all T's which is no longer a black hole solution. It is simply the Euclidean AdS space periodically identified, $\tau \sim \tau + \beta_1$. This space is called thermal AdS and will be denoted X_1. Note that S^1_τ is no longer contractable. In order to be able to compare to X_2 we must have the same anti-periodic boundary conditions for the fermions (we did not have the choice of periodic fermions for X_1 because S'_τ was contractable.)

X_1 and X_2 have different topologies $S^1 \times B_4$ and $B_2 \times S^3$ respectively. Where B_n is an n dimensional ball with boundary S_{n-1}. This situation is nicely represented by tents in Fig. 6.

In order to compare the actions of the two solutions we must make sure we have the same TR_{S_3} for both X_1 and X_2 at the uv cutoff, $r = r_\infty$. This condition defines β_1 in X_1 in terms of T from

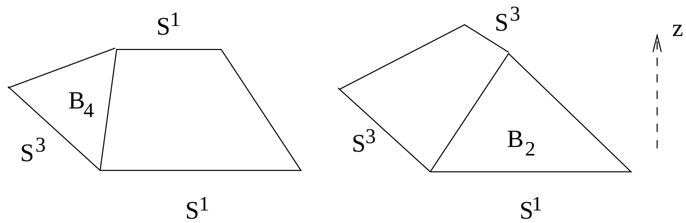

Figure 6: Left: Thermal AdS (X_1) which is topologically $S^1 \times B_4$. Right: AdSBH (X_2) which is $B_2 \times S^3$.

X_2.

$$\sqrt{g^{(1)}_{\tau\tau}}\beta_1 = \sqrt{g^{(2)}_{\tau\tau}}\beta_2 \tag{20}$$

$$\sqrt{1+\frac{r_\infty^2}{L^2}}\beta_1 = \sqrt{1+\frac{r_\infty^2}{L^2}-\frac{\mu}{r_\infty^2}}1/T \tag{21}$$

Now recall the central formula of the AdS/CFT correspondence relating the partition function of the CFT to the string partition function,

$$Z_{CFT}(S^3 \times S^1_\tau) \approx \sum_{\text{saddles}_i} e^{-I_{\text{sugra}}(X_i)} \tag{22}$$

which for large $1/G_N$ is dominated by the saddle point X_i with the smallest action (free energy.) The actions are infinite and require some sort of renormalization. We achieve this by taking the different in actions of the two spaces,

$$\Delta I = I(X_2) - I(X_1) = \lim_{r_\infty \to \infty} \frac{1}{2\pi G_N L^2}\left(V_{r_\infty}(X_2) - V_{r_\infty}(X_1)\right) \tag{23}$$

Because $R_{\mu\nu} \propto g_{\mu\nu}$ for these spaces the action is simply proportional to V_{r_∞} the regulated volume of the space. The difference in action is finite in the limit where we take away the cutoff $r_\infty \to \infty$.

$$V_{r_\infty}(X_2) = \int_0^{1/T} d\tau \int_{r=r_+}^{r=r_\infty} dr r^3 d\omega_3 = \frac{\pi^3}{4}1/T(r_\infty^4 - r_+^4) \tag{24}$$

$$V_{r_\infty}(X_1) = \int_0^{\beta_1} d\tau \int_{r=0}^{r=r_\infty} dr r^3 d\omega_3 = \frac{\pi^3}{4}\beta_1 r_\infty^4 \tag{25}$$

$$\tag{26}$$

Then one finds,

$$\Delta I = \frac{\pi^3}{8G_N}\frac{r_+^3(L^2 - r_+^2)}{2r_+^2 + L^2} \tag{27}$$

From which we conclude,

- $r_+ < L$ (low-T): $\Delta I > 0$, such that thermal AdS (X_1) has the smaller action. The free energy which follows from the CFT partition function is,

$$F \propto 0 \cdot N^2 + \mathcal{O}(N^0) \tag{28}$$

This conclusion requires a little more careful analysis of the local renormalization counter-terms, used in order to regulate $I(X_1)$. However because the volume $V_{r_\infty}(X_1)$ only depends on the upper radius r_∞ it should be completely removed by such counter terms. Then $Z_{CFT} \sim e^{-0/G_N}$ from which (28) follows.

The Polyakov order parameter for confinement satisfies: $\langle|\text{trU}|\rangle_{X_1} = 0$ since S^1_τ is not contractable. Both the above results indicate that the low-T phase is *confined*.

- $r_+ > L$ (high-T): $\Delta I < 0$, so the large BH solution dominates. In this case the free energy is, $F \propto N^2$. The Polyakov order parameter for confinement satisfies: $\langle|\text{trU}|\rangle_{X_2} \neq 0$ since now S^1_τ is contractable. Indicating a *deconfined* phase at hight T.

2.3 Cartoon of the thermal history of AdS

Using $x = \langle|\text{tr}U|\rangle$ as our order parameter we can sketch the free energy as a function of x for various temperatures. (Note this is a sketch because we don't know F or x away from the saddle points that we have computed above.)

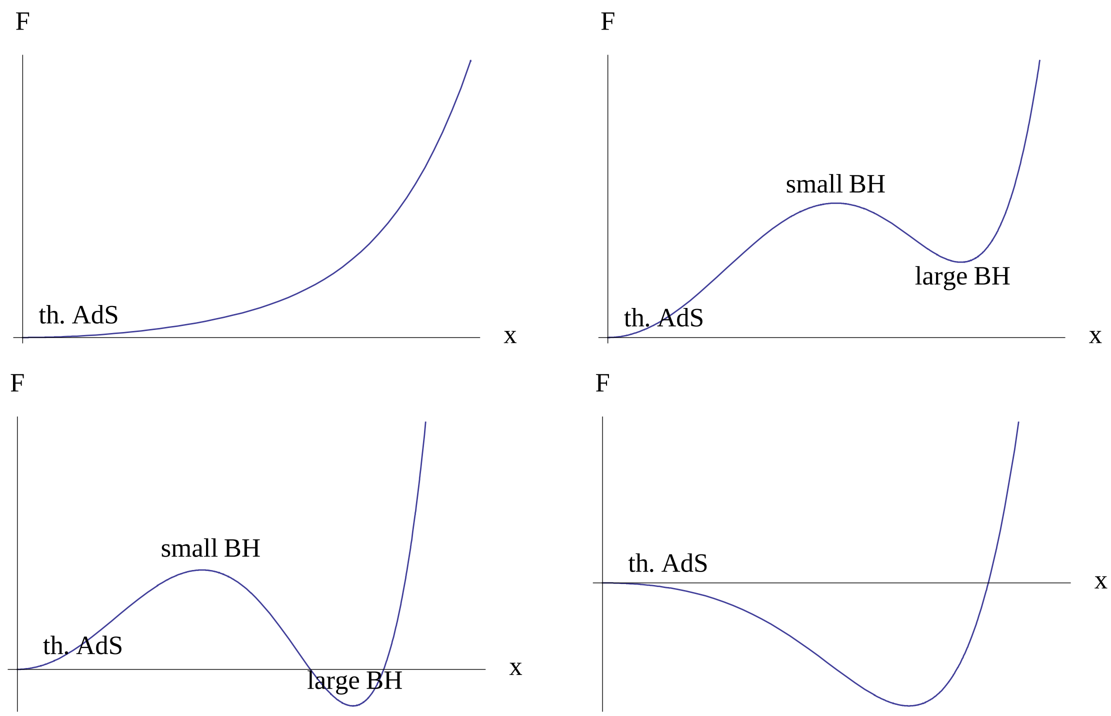

Figure 7: From left to right, top to bottom: $(T < T_1)$ Thermal AdS is the only saddle point. $(T_1 < T < T_{HP})$ At $T = T_1$ two new solutions appear corresponding to the small and large BHs. The local maximum is unstable and corresponds to the small BH. One can check that $x_{\text{small}} < x_{\text{big}}$ and $F_{\text{small}} > F_{\text{big}}$ as suggested by this graph. $(T_{HP} < T < T_{Hag})$ At $T = T_{HP}$ the large BH wins thermodynamically over thermal AdS. $(T > T_{hag})$ String theory on thermal AdS becomes unstable: $\partial_x^2 F(x = 0) < 0$. This is due to a tachyon in the spectrum of a closed string wound around the thermal circle: $\alpha' m^2 = -1 + N_{\text{osc}} + 1/(T^2\alpha')$ which gives a Hagedorn temperature $T_{Hag} = 1/\sqrt{\alpha'} \sim \lambda^{1/4} T_{HP}$.

References

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