IAIN STEWART: All right, so last time, we were talking about degrees of freedom in SCET. And I drew these two pictures and emphasized to you that the difference between these two pictures is really where the degrees of freedom live in momentum space. So this is the P minus P plus plane.

And unlike most effective field theories, in this effective field theory we have degrees of freedom that have to be identified by two variables rather than just one. So we need two variables to say where collinears are or where softs are. And there was generically, I said, a dependence on the process.

So SCET, or what degrees of freedom you need, depends what you're studying. And it's generically not the case that these two effective theories that I wrote down for you will describe every possible process. In fact, SCET, of course, won't describe every possible process.

So what effective field theory you need depends on what you're looking at. It just turns out that these two sets of degrees of freedom that I wrote down to you do describe a lot of different processes. And that's why the language of SCETI and SCETII is useful, but there's no guarantees.

You could run into a process that requires more degrees of freedom. And there are examples of processes that require more degrees of freedom than this, but this is still a useful starting point. And this is where we'll start.

And then we were talking a little bit about fields last time. And one thing I derived for you was that the collinear field has a power counting that makes it scale like lambda. This is the collinear quark.

And then at the end of lecture, we also derived a formula for the collinear gluons. So if we write down components of the collinear gluon plus, minus, and perp-- we remember what these notation means. This means n dot A. And this means n bar dot A.

And we figured out that the scaling for these guys is lambda squared 1 lambda. And we said that that was nice because we could put them together with a derivative and have a covariant derivative, which I wrote in terms of momenta just to sort of emphasize. So, if you like, this is momentum space, that these guys had the same power counting in the different components.

And so when we come and talk about gauge invariance, this fact that they have the same power counting in the different components will, of course, be important. Because what gauge invariance does is ties momenta to gluons. And if we didn't have the same power counting in the different components, then we would not have a gauge field corresponding to a certain momenta. OK, so another way of arguing that they should have the same power counting would have been directly from gauge invariance.

All right, I'd like to continue there. The one thing that you'll note, which is unusual from an effective theory point of view, is there's a set of fields that have no suppression at all. These A n minus gluon fields just scale like one.

So when you have dimensional power counting, all the fields have dimensions. You add more fields. You get higher dimensions. You get suppression.

Here, with this lambda power counting, there's a component of the gauge field that has no suppression. We can add as many of them as we want. And that costs nothing.

So let's do that and see what happens. So a priori, you might think that this is really bad because you could have an infinite number of different operators at lowest order just with a different number of these A n minus fields. And we'll see that actually things are not so bad, but I think it's best to do that by way of example.

So we'll consider an example that just has one collinear particle in it for simplicity. And we can do that by having a heavy particle that carries a lot of energy decay to a light particle, like a B quark decaying to an up quark. So in QCD, our current would just be u bar gamma b.

Gamma would be left-handed current. And we can consider the B quark to be in HQET. And we have an energetic u quark as what we want to consider.

It's not the only possibility, but we could pump all the momentum out through the current into the leptons into the e nu. But let's pump energy into the up quark, so that it's energetic. So here's the current in QCD. b goes to u.

And the notation that I'll use for this is a double line for the heavy quark and then a dashed line for the collinear quark. So this guy is going to be described by an hv field. This guy is going to be described by Cn field. This is production, so this will be a Cn bar.

So I'll reserve this notation of dashed lines for collinear quark propagators. OK, so the simple minded thing is just to write down an effective theory current where I replace the full theory fields by the effective theory fields. I replace the up quark, an up quark that's a Cn collinear field in some direction. And I replace the heavy quark by an hv for the b quark.

Then, effectively, that would be the tree level matching between this diagram and that diagram or the tree level matching from that particular diagram. But let's consider adding these A n minus gluons, since they were things that we could add to the current. So we can add any number of A n minuses here, and we'd have something that's the same order in the power counting.

So let's consider another diagram in QCD, where I just attach an A n minus gluon. And I'll attach it to the heavy quark. So I have a different number of external fields, but this one costs me nothing when I go to the effective theory.

Let's call this momentum here k. We'll call this guy here q. And this guy here we'll just say is MbV. So then k is MbV plus q.

And I can write q out in components since it's a collinear particle. And if I want to identify the big piece, then that would be the n bar dot qp's. So remember last time, from looking at coordinates, we can write out any vector in terms of the coordinates. And it's useful to do that since the coordinates have a different power counter.

Now, what's going to appear in the propagator here is k squared minus m squared minus Mb squared. So let's look at k squared. So if I square this term, I just get Mb squared. If I square this term, I get 0, but then there's a cross term as well.

And then there's some dots. And the dots are indicating terms that are suppressed. And so if I look at k squared minus M squared, then that's n dot v Mb n bar dot q. And this order lambda to the 0 since the momentum n bar dot q was order lambda to the 0.

So there's no power counting to the propagator. Another way of saying it is this propagator is off-shell by a hard amount. These scales are hard. And if I go back to my picture, that means they're up here in this purple region, purple box around this.

OK, so the degree of freedom, which is this purple propagator is actually living up in that purple region. It's something we have to integrate out, OK? So let's do that.

So integrating it out just means expanding the diagram. And I've already expanded the propagator, so there's the denominators. We just have to expand the numerator.

OK. Let me write out A n mu, but we replace the spinner by the field. And I'll start by just writing in the full propagator, and then we'll start expanding it. I'm using a convention here for the gluon interaction in QCD, which is plus igTA gamma mu.

So there's the full theory. All I've done is take the full theory. And rather than write spinners and a polarization vector, I've just written the fields. Because I'm, in the end, interested in looking at what kind of operator I get.

So I'm interested here in taking the piece, which is not suppressed. So let's just do this slowly. So there's an index A on this guy as well.

So let's write out the leading order piece of the numerator. I've collected some i's, put the M, the g out front. So if I take all the order 1 pieces of the numerator, then these are the order 1 pieces.

Mb is order 1. k I can replace by MbV plus q. And then there's an MbV slash that's order 1 and an n slash over 2 n bar dot q. That's the same decomposition we had over there.

Here, because I'm interested in certain polarization, let me pull out that polarization. Then I have my denominator. And my denominator we already expanded, so let's just write that in.

OK, so far, so good? So this piece here is 0 because n squared is 0. n slash squared is n squared. So then we have this piece.

For this piece, 1 plus v slash, remember, is part of a projector that could act on the heavy quark. But there's this n slash in the way, so we want to push it through. So we can do that.

So let me write out the two pieces from pushing it through. There's an Mb that cancels this Mb once that term's gone. So let me do that. When I push it through, I switch to 1 minus v slash.

There's still a 1 over n dot v here. And then there's an hv. And 1 minus v slash on the heavy quark field is 0. And then the n dot v cancels n dot v.

So if I want to write this putting, now, this A n field inside because the TA is inside, then I would write it like this. So the momentum here n bar dot q is order 1. And this n bar dot A n is order 1. So this operator is the same size as this operator here.

And that's what I was saying that we should worry about, the fact that we can add these gluons without any cost. And we just added one and found out we have an operative that's the same order. So the way you might draw this-- and there's a convention also for collinear gluons where you put a line through them, so this is a collinear gluon-- is that this guy is the same order as this guy, OK? Well, we'll come back to this, talk more about it in a minute.

Let's consider another diagram. We attached the gluon on the left. Let's also attach it on the right, see what happens.

So again-- QCD diagram, doing the same thing, attaching the gluon here, some q. This is some momentum p. And this is p minus q.

This case is different than the previous case. The reason that this case is different is that both p and q are collinear. And so p minus q is collinear, too, same scaling as p and q.

So this propagator is not off-shell. In our hyperbolas, it would live on the blue hyperbola that I drew, which is well within the effective theory, not outside the effective theory. So we don't want to integrate out this propagator here. And what that means is that there will be, in the effective theory itself, an interaction, a Lagrangian interaction that takes into account this diagram.

So in the effective theory, if you like, if I draw an effective theory diagram, there will be an effective theory diagram that allows us to attach that gluon because this propagator here is a propagator in the effective theory. Unlike the one that I just erased which was off-shell and had to be integrated out of the effective theory, this one is inside the effective theory, OK?

So adding on the left does something different than adding on the right. On the left, we've knocked the quark off-shell. We add on the right, it's close to mass shell.

So we can consider generalizing what we just did by adding more gluons. Remember that we don't have any restrictions on how many we can add. So let's just consider adding more.

Let's add m of them. And I'll call the momenta q1, q2, to qm. And all these propagators here are off-shell.

This is a QCD graph. When I calculate graphs like this, if I have these gluons there, I have to also consider cross-diagrams. So rather than trying to draw cross-diagrams, let me just cross-gluon graphs.

And what you would expect from what we've said before is that in the effective theory this is going to catch on to some kind of Feynman rule that looks like this, but then has a bunch of gluons that can come out of the vertex, in this case m of them. That's what we expect. And so we expect a generalization of this operator with m fields rather than just 1.

So if we do this calculation, it actually turns out to not be too complicated. I can include all cross-diagrams by sum over permutations. I have one of these n bar dot A gluons for each external gluon line with a corresponding color factor.

And I keep the largest piece of the denominator. And I'm always just getting the order 1 momenta. And there's a slight complication that, when I have a graph like this, if I look at the momentum of this, it's q1. But if I look at the momentum of the next one, this would be q1 plus q2. And so the denominators here are not just simply q's, but sums of q's.

OK. And these n's would be dotted into fields. And T's would be dotted into fields. So if we want to write this as an effective theory operator, then we should think about what this vertex means in the effective theory.

And we should be a little bit careful. Because what this means, if you think about it in terms of the locality of this diagram, is that all these gluons are coming out of the same point. So it's like phi 4 theory where you divide by 4 factorial.

You have to be a little bit careful because any one of these gluons, from the point of view of the external states, could be as equally good as an attachment. So when we put in the gluon fields, we ought to be careful about those factors. So this is the right Feynman rule.

And one way of just thinking about it is we'll have m of these fields, but then we'll divide by an m factorial. So we can write down the complete result for the two level matching as an operator since we just calculated all the diagrams.

So we started with this current. We're in QCD. And we matched it at tree level onto a current that I can write like this, hiding all the complexity in this thing I call W. I have to sum over however many attachments, which I was previously calling m, but now I'm calling k.

And if you study this guy for a while, actually what is true is that this guy has a name. This is a momentum space Wilson line. So what is a position space Wilson line? That's where it really looks more like a line.

So a Wilson line is a string of gauge fields that goes between two points, here between minus infinity and 0. There are some ordering to the fields because they don't commute. Because they're not abelian.

And the fact that it's a line means that we go in a straight line between the two points. So here's a path which is a straight line s between minus infinity and 0. And actually, this formula here, the Fourier transform of this, is giving you the Wilson line that I drew up-- the Feynman rule that I drew up there or the fields that I drew up there.

So there's some ordering to the fields. And what this P does is it orders the fields in an appropriate way. Namely, it puts the guys with the larger argument to the left.

So on your homework, you're going to do the exercise of thinking about doing this matching for two gluons and taking the Fourier transform this and connecting these. At least I believe that's what I asked. That's a very good exercise to see everything work out that I'm just describing to you here in words.

So one way of thinking about what's happened in this effective theory-- and this actually turns out to be generically true-- is that rather than having this embarked on a field, which was order 1, it actually can be traded for this Wilson line object. So I just showed you that that happened for this particular example, but it turns out actually to be genetic, that, instead of talking about the n bar of A field, we can talk about this function of the field, which is this Wilson line.

Lift that for a sec. So this is true in this operator. We just got this Wilson line. In terms of the Wilson line, things are pretty simple. And it actually turns out to be generically true, and we'll see that later on.

And we'll also talk about gauge invariance later on, which has to do with this story. It turns out that this Wilson line can also be understood from the point of view of gauge invariance. The need for this Wilson line can be understood from the point of view of gauge invariance.

And I can at least describe that to you in words. So if you look at this diagram here, we can't attach collinear gluons to this quark. But this carries color.

We can attach collinear gluons to this quark, no problem. It doesn't knock it off-shell or anything. So we're going to have a problem with gauge symmetry because we have two things that are colored. A priori, we're going to have a problem with gauge symmetry because we have two things of color, one of which we can't attach effective theory fields to and one of which we can.

But there's also operators with gluons in them, which are these ones. And what effectively happens is that the gauge transformation that used to be for this heavy quark field, it gets moved into this Wilson line. So this Wilson line will transform in a way that compensates for the transformation of this collinear quark field.

And that's exactly because the gluons here, which were kind of the corresponding gluons in QCD for the gauge transformation, are explicitly integrated out to give that Wilson line. So it's all tied together. We'll come back and talk about that in more detail later.

All right, so let's consider now the effective theory now that we have some idea of how these calculations are working. And let's consider this SCETI effective theory. So we have a collinear mode and an ultra soft mode with momenta that we're scaling in this way.

And we also know, if we consider the Feynman rules, we can look at what the propagators would look like. So let's consider, first of all, having a collinear gluon. And let's just try to think about what kind of propagators the effective theory is going to have.

So maybe this comes from a heavy quark. So we have this possibility of having a collinear gluon, momentum q. We expanded the propagator for that last time.

And the important point about this being a collinear gluon is that p and q are the same size. So we never drop p relative to q or q relative to p. So if we write out the components, we did this last time. And we found that there was really no expansion in the denominator. And in the numerator, we could make an expansion and just keep the leading order piece.

So the propagator would be proportional to this. There's some spin structure as well that I'm not writing. But q of the gluon is of order p of the quark. So nothing gets dropped.

So this is going to be a propagator that exists in the effective theory. That's what I was saying before. The other thing we could do is we could attach an ultra soft particle.

We could actually attach it to either side, but let me focus on the right side. So there's going to be some ultra soft gluon. It's not collinear to any direction. Let's call its momenta k, call this still p.

Now, k is much smaller than p by our power counting. So the k is of order lambda squared. So in n bar dot k is much less than n bar dot p, which is of order lambda 0. k perp is much less than p perp, which is of order lambda.

And then the n dot k is or order n dot p because those are both lambda squared. So here it's not going to be like over there. There is actually going to be an expansion that occurs because things are of different size.

So if I only keep the leading order terms and the propagator here, well, the numerator I just have n bar dot p because n bar dot k is small. And the only place that I need to keep both is in this n dot term. So there is an expansion that goes on.

And again, if I look at how off-shell this propagator is, it's perfectly within the effective theory. So this is a propagator with off-shellness that leaves it inside the effective theory. OK. So it's not far off-shell by any means. In fact, it's just as off-shell as the propagator over there for the collinear particles.

So it doesn't really make sense to think about treating this propagator any different than treating this propagator from an effective theory point of view. But you see that the Feynman rule has to somehow know whether you're in this situation where you need to keep the momentum of the gluon or whether you're in this situation where the momentum should be expanded. And we'll see how that works.

So this is by way of sort of giving you a hint as to what's to come and what kind of complications we'll face in the effective theory because the effective theory can interact with collinear gluons or these ultra soft gluons. And it somehow needs to be only keeping leading order terms. If it's really the leading order effective theory, then we should just keep this term. We shouldn't keep these higher order terms. But over here, we have to keep all the terms.

So what should the leading order effective theory do? What should we demand of it? We've basically, now, set up all the things that we need to think about in order to figure out what the leading order effective theory is. So let's do that.

OK, so what do we demand of this Lagrangian? It should yield the propagator of course and, also, interactions. Within this effective theory, it has interactions both with collinear gluons in that diagram over there and with ultra soft gluons.

And you can see even from here why we need two different fields because there has to be some way of telling that we have a different propagator in this case than in that case. Or put another way, the external field here has a different power counting than the external field here. So already we're kind of seeing that we need two different gluon fields in this effective theory, as we said earlier.

So the Lagrangian has to have interactions with both of them. It has to have both quarks and antiquarks because remember that collinear quarks and collinear antiquarks both existed as leading order things. That's not like HQET where one of them gets integrated out.

And basically, that is just the statement that, say you have a collinear gluon going along very quickly. It can para- - create a quark and an antiquark. And if they're, again, collinear, that's something that has to be allowed to happen at order 1 in this effective theory.

OK. So, so far, so good. This is just saying, what kind of fields are we going to have in this Lagrangian. This third bullet is related to what I was just stressing, that we have to get a leading order propagator in different situations.

So somehow the effective theory has to know something about the size of momenta that are running through it. And if we want to be strict about defining the effective theory and just defining the leading order term, it's not OK that we expand later. We really have to expand ahead of time.

And so whatever the Lagrangian is, it should just give this. And it should just give this. And it should not give these dots, OK?

So that's what I mean by having no additional expansion. And the final thing is a little more subtle. And that is that we should think ahead.

So when we design this effective theory, we're not only going to want to describe the leading order term, but we're also going to want to describe power corrections. Because if we can't describe the power corrections, we don't really have an effective field theory. So whatever we do, we should make sure that we have a notion of what's going on with higher order terms, that they're well-defined things, that we can think ahead about what the operators are going to look like.

And we need that in order to know that we can formulate those operators and make sure they're suppressed. So another way of putting it is that we should set things up, so we don't have to re-set things up when we start talking about power corrections.

So we really should think about the effective theory globally even though we're sort of starting by formulating a leading order term. We should think about the power corrections as well and how those interactions are going to behave. OK, so that's saying that we want to think about, at least, what these dots are going to look like, what this plus dot, dot, dot is going to look like. OK, so this is our goal. And we'll go through it slowly.

So this is a top-down effective theory. So we can start with QCD, and then we can integrate out the off-shell degrees of freedom. We can just formulate from what SCET is by starting with the QCD Lagrangian and splitting it into the fields that we want and manipulating it.

So let's start with psi bar iD slash psi. And let's write psi in terms of two fields. And this goes back to our discussion when we were talking about spinners.

So these two fields have a projector on the full theory field, and they obey n bar slash, n slash, n slash, n bar slash. Cn is Cn. And the n bar slash n slash by n bar is by n bar. So I want to write out the QCD Lagrangian in terms of component because the different derivatives, the different components of the covariant derivative and the different regular derivatives inside that derivative, have different power counting, OK?

So let's do that. And I'll introduce these two fields in order to do that. So if you like, I'm just taking QCD and writing it in these coordinates that we're setting our usual coordinates for talking about SCET.

So decompose the D slash and just write out the field as these two pieces. And then I can multiply these out. And because of the projection relation, some of these products will vanish. So if I only write the pieces that don't vanish, then I only have four pieces, which are these four.

So all other combinations that I didn't write vanish, and those four survive. We'll do one example, so you see what the strategy is for getting rid of the other terms.

So let's consider one of the other terms. Let's consider the guy with the D perp slash, but between two Cn fields. So I can insert the projector. And then I can move it through the D perp slash.

I'm moving two things through, and they both have 0 dot product with D perp. So there's no additional terms. I just can push it through. And n slash on Cn bar is 0.

So n slash on Cn was 0, remember. And if you take the dagger and make it into a bar relation, you also have this formula. So this is 0.

And a similar thing-- by inserting projectors, we can figure out which terms are non-0. And these ones I'm writing are the non-0 ones. This guy's 0. That's why I didn't write it.

OK, so any questions so far? Hot today, drink lots of water. OK. This is just QCD. I haven't done anything. I just write QCD out and assign some strange coordinates.

So now, I'm going to do one thing, which is going to end up simplifying our lives a little bit later on. And I'm going to use the fact that, when we talked about production of quarks, we said we're going to produce this Cn type of quark and not the phi n bar bar type of quark.

So phi n bar, remember, corresponded to sub-leading spinner components. So let's just decree that I don't really care about this field. And, therefore, I don't need to have any current in my path integral that I couple to this guy.

So it's like an auxiliary field if you like. And because the path integral in this fermionic field is quadratic, we can just remove it from the path, just do the path integral over it. Once we know that there's no source term for it, we can just do the path integral since all the terms are explicitly written over there.

So let's see what happens when we do that. So doing the path integral for this quadratic field just means solving the equations of motion. So we take the variational derivative with respect to phi n bar of the Lagrangian. And that gives us the following equation.

We could multiply both terms in this equation by an n bar slash over 2. And then we can use the projection relation. Just to simplify the Dirac structure a little, we can move the Dirac structure from here to here.

And then we could formally solve this equation for the phi n bar. It's an inverse covariant derivative. I can push this through this at the cost of a sign.

So if I move it to the right-hand side of the equation, that takes care of that sign. So I did two sign changes. I moved something to the right. And then I pushed these through each other to get that equation.

OK. So what this is saying, in terms of the original field, is that I have this equation for psi. So if we take that result and we plug it back into our Lagrangian, which I meant to give a star, then we'll just have an equation in terms of the C.

We already used two terms. And so those two terms, when you plug it back in, they just cancel. So there was two terms in the Lagrangian that had psi n bar bar.

And the remaining two you have that.

OK, so this is Lagrangian that's just in terms of the Cn field. We'll call it double star. Previous guy was supposed to be star.

OK, so several questions arise. What have we done? Do we like this inverse covariant derivative? Are we happy with that? Are we unhappy with it?

So what does an inverse covariant derivative mean? Well, actually just what is an inverse derivative mean? This is an inverse operator.

So what does that mean? So the way you define an inverse operator is exactly how you define an inverse operator in quantum mechanics. So let me remind you of that. This is the analog of, in quantum mechanics, having, say, a 1 over r potential.

But if r is an operator, then you have a 1 over an operator. And the way that you define that is by the eigenvalues.

So 1 over i in r dot partial of some field i of x you can write it out in terms of momentum space. So if we go over to momentum space, that's the eigenbasis. And then the eigenvalue is just 1 over the momentum, this component of the momentum.

OK, so it's just the Fourier transform of that. So this is [INAUDIBLE]. Yup.

AUDIENCE: Yeah. I have a question about whether or not you've actually done nothing.

IAIN STEWART: I have done nothing, yeah.

AUDIENCE: Yeah, it seems [INAUDIBLE].

IAIN STEWART: Yeah, that's my next comment.

AUDIENCE: Right. And so phi n bar--

IAIN STEWART: Yeah, so--

AUDIENCE: Actually corresponding to this other new spinner would mean that you're only considering interactions that are all collinear. In the original piece, the [INAUDIBLE] does contain that interaction.

IAIN STEWART: The only difference between what I've done so far in QCD is that, in QCD, if we were to think about QCD, we'd couple external currents both to this phi n bar as well as the Cn. And that's the only difference.

But there's a small community of people that actually think about this Lagrangian as the QCD Lagrangian. And that's because, if I was willing to a couple a current to this particular combination of fields, I could still call it QCD, OK? So in that sense, if I really demand that I'm only producing Cn's and I just have a term that's, like, JCn and I don't have this phi n bar, then it's something different because I'm not directly able to produce these guys.

But if I allow myself, also, to couple to this combination here, then I can do everything in terms of these Cn fields. And it's really just QCD.

AUDIENCE: So star star is exactly collinear QCD if you're only producing in--

IAIN STEWART: Exactly.

AUDIENCE: --forward direction or something? [INAUDIBLE]

IAIN STEWART: Just even simpler, it's exactly QCD if I only produce these Cn. If I just say I'm only interested in operators that produce this particular spin component--

AUDIENCE: But if you try to say that--

IAIN STEWART: Yeah.

AUDIENCE: [INAUDIBLE]

IAIN STEWART: No, no. I haven't I haven't made any expansion to make it collinear yet.

AUDIENCE: But if there was a hard scatter with two totally different directions [INAUDIBLE], you could produce phi n bars, right?

IAIN STEWART: Yeah, but that would be another Lagrangian. So this Lagrangian, in the end, will only be useful for producing particles in the n collinear direction. That's what I'm going to be after.

And I'd have another copy of this. For each linear direction, I have one additional copy of what I'm doing now.

AUDIENCE: Right, so you've lost the ability to produce hard scattering by doing this.

IAIN STEWART: By just doing this.

AUDIENCE: But as long as you don't talk about that, it's still [INAUDIBLE].

IAIN STEWART: Right. That's right. OK. So we're far from done. And really this is just sort of a warm up. Really what we've done here is we've just written things out in terms of components. And it's true that these derivatives here have different power counting.

And we're going to make use of that in a second. And we've also just sort of focused our attention on this guy that we could produce at leading order from a hard scattering in some particular direction, such as in our b to u example. So if we're interested in just one collinear direction, then there's two components of the spinner that are always the big components. And we focused on a field that gives those components directly.

So what do we have to do still? So there's three more steps. So we have to separate collinear. So far, we just have one type of covariant derivative.

We have to separate out the fields that are in that covariant derivative. In particular, the ultra soft and collinear gauge fields have to be separated out. Another thing we have to do is we have to separate the collinear and ultra soft momenta.

There's two different types of momenta, as we saw in this simple example of a gluon momenta flowing into a propagator. And we actually have to distinguish them. And then we have to expand.

So until we do these first two steps, there's not really anything to expand. And so once we've distinguished those, we'll be able to say that certain things are larger than others. And then we'll be able to figure out what the leading or Lagrangian actually is.

OK, so let's start with this second step. So this is step two. So remember what the power counting was. The collinear gluons had this scaling. The ultra soft gluons had this scaling, which is the same as an ultra soft momenta.

And the right way of doing this kind of thing would be to take Feynman diagrams and do some matching calculations, but sometimes we're able to get away with just thinking about doing things by writing down field relations. And that's almost true here. When we were drawing our Feynman diagrams before, if you think about one gluon, this is certainly true.

So we drew Feynman diagrams before. We had this one, and we had this one, ultra soft collinear. And we just consider one at a time. So we could just think of adding them.

And that's kind of almost true. It turns out that it's true up to some terms that we don't care about for the moment, OK? But these terms are power suppressed.

And we'll see where they come from later on. So the way you can think about this, if you ignore that complication, is that you really just divide the gauge field into two pieces. There's one place that you've perhaps seen that before, when you talk about a background gauge field, right?

You say you have a quantum field and a background field. And that's very nice if you go to background field gauge. And that's kind of physically what's going on here because we have two different modes that have different wavelengths.

These guys are really long wavelength. These guys are shorter wavelength. Remember that the p squared of these guys is much larger than the p squared of these guys.

And think about this formula in that way, that this is like a classical background field to the Cn and the A n mu since the p ultra soft squared had a scaling which was lambda to the fourth lambda squared squared. And that's much smaller than p collinear squared, which was q squared lambda squared.

So that's saying that this guy has much larger wavelength, a long wavelength mode to these short distance modes. So we can think of this as a slowly varying classical background. And that's one way of thinking about this formula. We'll exploit that a little bit later on as well.

So now that we have the sum of them, we can just think about how big one is relative to the other. And actually, the coefficient that's between them doesn't really matter too much for that. So let's think about the power counting I already wrote on the board.

So the power counting is that n bar dot A n is order lambda to the 0 and so is much greater than the n bar dot A ultra soft, which is order lambda squared. Same for A perp n, that's of order lambda. That's much greater than A perp ultra soft.

And so those two formulas mean that it would never have a comparison between this and this. I can always drop the smaller one. And just like our momentum, there's one case where that's not true where we have to keep both of them. And that's for the n dot A component.

So here's something we can drop, n bar dot A ultra soft and A perp ultra soft. That leading order can be dropped, OK? So that's what we're going to do and step two.

Step three corresponds to dropping momentum relative to another momentum. So there was an expansion of the propagator in the case where we had an ultra soft and collinear at the same time.

And let's just write that down again, but let's keep the next order term. And I'll ignore the numerator since just looking at the denominator will give us enough information for this discussion. We could do the numerator, too. It would just lead to more terms.

So there are some terms that I can expand. And then I get the propagator squared, or I get a P squared squared, because I expanded something in the denominator back into the numerator. So the next order term comes from a dot product in the perp momenta because the Pn perp is large.

And so we get a term with two k ultra soft dot Pn perp that corrects the leading order term. And that's the first sub-leading term. So if you think about the power counting of these, this guy is lambda to the minus 2.

And this guy is lambda to the minus 1. OK, so there has to be something in the effective theory that, at higher order, will reproduce this second guy.

There will be some power suppressed Feynman rule in SCET to reproduce this second term.

So what that means is that you can't just think that you ignore this guy completely, this k ultra soft perp. You might just say, well, I just set all the k ultra soft perps and the k ultra soft minuses to 0. And that would be something that you can get away with, if your careful, at lowest order. But at sub-leading order, you can't. You need to think that they exist, too.

So whatever formalism we develop should not set them to 0. And that was one of my bullets that you should think a little bit ahead. And this is me thinking ahead, that I'm not going to be able to just set those momentum exactly to 0. I'm going to need them when I start talking about sub-leading terms.

OK, so we expand just like we don't throw away the sub-leading gauge fields because we need those at sub-leading order as well. We neglect them in the leading order, but we have to sort of think that they exist still.

And with momenta, it's a little different since fields carry momenta. And so what you should think is that the fields are allowed to sort of carry these momenta, but the Feynman rules of the leading order theory are just not sensitive to them. And that's the right way of thinking.

OK, so what we need is an expansion of the theory that will do this. And that's called the multipole expansion.

So let me remind you of some things about multipole expansion. Well, a multipole expansion you see in electromagnetism, remember? You look at a charge distribution far away. You see the total interval over that charge distribution and then the dipole.

That's the usual thing that you think of when you think of the words multipole expansion. You think of E&M. And it's similar here, except we're doing it for fields.

So in position space, it's similar to this where we expand and we neglect some coordinates. So let me show you how a multipole expansion works in one dimension in position space. So let's consider the following. Consider an integral dx psi bar of x A of 0 psi bar of psi of x.

So one of the fields I stuck at 0 I expanded out the coordinate around 0, whereas the other ones I kept the x. Let's see what this does for the momentum space by Fourier transforming it-- so some momenta, some phases, and then some momentum space vehicles.

And I just put 0 because I put one of the fields in the coordinate space at 0. Do the x integral. So I got a delta function that says P1 is equal to P2 because I didn't have anything in the k.

And this is kind of along the lines of what we want. It says that the momentum P1 is equal to the momentum P2. And that's like expanding in k. So if we think about having some Feynman diagram where we have P1 and P2 and some k coming in, if we want in some component that P1 is equal to P2, then this is achieving that.

OK, so this is the analog of the kind of expanding coordinates that you do in E&M. Expanding into coordinates of a field is doing the same thing. It's giving a momentum conservation that ignores the momentum conjugate to that coordinate that I expand in.

So what would the next order term look like? So we're still here in one dimension. If you think about the expansion, the next order term would be a term like this, then at x equals 0. Sorry.

I don't really need dot since this is one dimension. So that would be the next term in a Taylor series of the field about x equals 0. And if you do the same thing, Fourier transform this guy, you can work out that you get the following.

OK, so the derivative here on the coordinate, that comes with k, momentum. Because I should Fourier transform the field, then take the derivative, then set the coordinate to 0, so I can still get a k. And this x actually just ends up leading to a delta prime.

You can write it as a derivative. And x can be written as a derivative in momentum space. And then it's a derivative on the delta function.

OK, so that's what the second order term would look like. So if you were to think like this, you have some kind of Feynman rule that's proportional to a delta prime. And then however you deal with a delta prime, if you're Feynman rule had a delta prime, what you would do is you would integrate by parts. And then the delta prime would start taking derivatives of other things.

So that is a possible route, but we're not going to take that route. We're going to do something simpler. So we could formulate the multipole expansion in position space like that, but we're actually going to formulate it in momentum space.

So we're going to do the same thing, but we're going to do it in momentum space. What are some reasons to do that? Well, when you do Feynman diagrams, you do them in momentum space.

So if we've already formulated things in momentum space, then we'll immediately get the Feynman rules we want to do loop integrals and stuff like that. So that's an advantage of the momentum space is that you more directly get the Feynman rules you want and the diagrams in the form you want.

It simplifies, actually, slightly the formulation or somewhat the formulation of gauge transformations, which we'll talk about later on. And what I think is a nice kind of advantage is where the momentum expansion sits is in the propagator, not in the vertices.

We don't get Feynman rules with delta primes, but we then have to integrate by parts to understand. We just immediately get insertions on propagators which would correspond to the derivatives that I would take from integrating that delta prime by parts.

So this term that we saw before that looked like a propagator squared-- which was when we were sitting in momentum space and we expanded the propagator to second order. So we had this term that was ultra soft perp dot Pn perp. And then it had this denominator that was squared.

If you ask where in the effective theory is that, what is going to be the thing that corresponds to that, it's going to correspond to some collinear propagator with an insertion on it. And the insertion gives this numerator. And then I have two propagators. And that gives the two denominators.

OK, so that's how that term is going to come about. So there will be some sub-leading Lagrangian that I could call L1 that I can insert on the propagator. And that would give rise to these terms that we were saying were higher order.

OK, so we have to figure out how to do this. And I will just give a little bit of a picture for it today, and then we'll continue with it next time. So I'm going to have to back up a bit.

So let's back up to the Lagrangian that we had before. And I want to just introduce, since we were not close to being done-- we had sort of started down the path, but we weren't done yet. Let me give a slightly different name to this field. Let me put a hat on it.

So I want to do the expansion in momentum space. So let me Fourier transform. So I'll call psi n twiddle of P, do the full Fourier transform of this guy, which was previously just a field x. And we'll think about doing expansions in this momentum space field.

And for now, I'm also going to make one other simplification. And that is to only consider the quark part. And we'll put the antiquarks back in later.

So if you like, we're considering sort of the a's and not the b's in the decomposition of the field. OK, so let's think about this and think about how we can do an expansion that would correspond to this multipole expansion. And the analogy that I'm going to exploit here, although it's not perfect, is the expansion that we did in HQET.

So in HQET, we said we had a full momentum for the b quark. And we split it into a piece, which is kind of the big piece. And then we had a piece that was the small piece.

And I'm going to do something similar in SCET. I'm going to say I have a full momentum-- I'm going to move the indices up-- that I'm going to split into two pieces because of the power counting, a big piece and a small piece. The big piece here will be the terms which were of order 1 or lambda. And then the rest here will be something that's of order lambda squared.

And I allow that residual piece not just to be in the plus direction, but to be in any direction. Because remember, at sub-leading orders, I'm going to have to care about the other components as well. So let me just say that the residual could have order lambda squared pieces in any component.

In the minus and the perp, I'm going to expand with these residuals being smaller than these terms here. And this momentum I'm going to call a label momentum. And this momentum I'm going to call residual momentum.

We'll see why that's a good language next time, but it's kind of connected to the idea that, in the HQET, we labeled the fields by this V. And we're going to see that, for at least some of the interactions, it's a reasonable way of thinking. These large momentum is just labels on the fields.

OK, so we'll talk more about that next time. I think I don't want to go further with it. But basically, we're going to make an expansion in residuals being smaller than these guys. And in order to make that expansion, I have to have some notation for the different pieces.

And we'll see where that leads us next time. It'll effectively lead us to the same thing as this multipole expansion in position space, but it's a little nicer, I claim. I like it a little more. Questions? Good.