then we may move all usoft wilson lines into the usoft part of the operator yielding

$$Q_{\mathbf{8}}^{1,5} = [\bar{h}_v^{(c)} \Gamma_h^{1,5} Y T^a Y^\dagger h_v^{(b)}]\, [\bar{\xi}_{n,p}^{(d)}\, W \Gamma_l C_8(\overline{\mathcal{P}}_+) T^a W^\dagger \xi_{n,p}^{(u)}]. \tag{11.8}$$

Matching this SCET$_{\rm I}$ result onto SCET$_{\rm II}$ by the replacements $Y \to S$ and $\xi^{(0)} \to \xi$, $W^{(0)} \to W$, we have

$$Q_{\mathbf{0}}^{1,5} = [\bar{h}_v^{(c)} \Gamma_h^{1,5} h_v^{(b)}]\, [\bar{\xi}_{n,p}^{(d)}\, W \Gamma_l C_0(\overline{\mathcal{P}}_+) W^\dagger \xi_{n,p}^{(u)}] \tag{11.9}$$

$$Q_{\mathbf{8}}^{1,5} = [\bar{h}_v^{(c)} \Gamma_h^{1,5} Y T^a Y^\dagger h_v^{(b)}]\, [\bar{\xi}_{n,p}^{(d)}\, W \Gamma_l C_8(\overline{\mathcal{P}}_+) T^a W^\dagger \xi_{n,p}^{(u)}]. \tag{11.10}$$

Now, taking the matrix elements between the appropriate hadronic states we have

$$\left\langle \pi_n^- \right| \bar{\xi}_n W \Gamma C_0(\overline{\mathcal{P}}_+) W^\dagger \xi_n \left| 0 \right\rangle = \frac{i}{2} f_\pi E_\pi \int_0^1 dx C(2E_\pi(2x-1))\phi_\pi(x) \tag{11.11}$$

$$\left\langle D_{v'} \pi_n^- \right| \bar{h}_{v'} \Gamma h_v \left| B \right\rangle = N' \xi(\omega_0, \mu). \tag{11.12}$$

We are able to achieve this factorization because with B, D purely soft and π purely collinear there are no contractions between soft and collinear fields. So we find that our final factorization result is

$$\langle \pi D | H_W | B \rangle = i N \xi(\omega_0, \mu) \int_0^1 C(2E_\pi(2x-1), \mu)\phi_\pi(x, \mu) + O(\Lambda/Q) \tag{11.13}$$

where $\xi(\omega_0, \mu)$ is the Isgur-Wise function at maximum recoil and

$$\omega_0 = \frac{m_B^2 - m_D^2}{2m_B} \tag{11.14}$$

This result also applies to other B decays such as

$$\begin{aligned} &\overline{B}^0 \to D^+\pi^-, \quad \overline{B}^0 \to D^{*+}\pi^-, \quad \overline{B}^0 \to D^+\rho^- \\ &\overline{B}^- \to D^0\pi^-, \quad B^- \to D^{*0}\pi^-, \quad \overline{B}^0 \to D^+\rho^- \end{aligned}$$

11.3 Massive Gauge Boson Form Factor & Rapidity Divergences

11.4 p_T Distribution for Higgs Production & Jet Broadening

12 More SCET$_{\rm I}$ Applications

(ROUGH)

In this section we will apply the SCET formalism developed in previous sections to a few additional processes that either use SCET$_{\rm I}$ or a combination of both SCET$_{\rm I}$ and SCET$_{\rm II}$ (where the more complicated part of the factorization occurs within SCET$_{\rm I}$). In particular we will consider

- $B \to X_s \gamma$
- Drell-Yan $p\bar{p} \to l^+ l^- X$: inclusive, endpoint, and isolated factorization theorems

12.1 $B \to X_s\gamma$

(ROUGH) In this section we treat the incluzive weak radiative decay $B \to X_s\gamma$. This decay is defined by the effective Hamiltonian

$$\mathcal{H} = -\frac{4G_F}{\sqrt{2}} V_{tb}V_{ts}^* C_7 \mathcal{O}_7, \quad \mathcal{O}_7 = \frac{e}{16\pi^2} m_b \bar{s}\sigma_{\mu\nu}F^{\mu\nu}P_R b \tag{12.1}$$

with $F^{\mu\nu}$ the electromagnetic field tensor and $P_R = \frac{1}{2}(1+\gamma_5)$. The decay is defined such that the photon momentum is opposite the collinear jet i.e. $q_\mu = E_\gamma \bar{n}_\mu$.

The photon energy spectrum of the decay is

$$\frac{1}{\Gamma_0}\frac{d\Gamma}{dE_\gamma} = \frac{4E_\gamma}{m_b^3}\left(-\frac{1}{\pi}\right)\text{Im}T(E_\gamma) \tag{12.2}$$

where

$$T(E_\gamma) = \frac{i}{m_b}\int d^4x\, e^{-iqx}\left\langle \overline{B}_v \right| \text{T}J_\mu^\dagger(x)J^\mu(0)\left|\overline{B}_v\right\rangle \tag{12.3}$$

Is the forward scattering amplitude with EM current $J_\mu = \bar{s}i\sigma_{\mu\nu}q^\nu P_R b$.

We will consider the endpoint region of the decay in which nearly all of the final state energy is in the photon. Analyzing this process in the rest frame of B, we find that the final momentum X

$$p_X^\mu = p_B^\mu - q^\mu \tag{12.4}$$
$$= \frac{m_b}{2}(n^\mu + \bar{n}^\mu) - E_\gamma \bar{n}^\mu \tag{12.5}$$
$$= m_b\frac{\bar{n}^\mu}{2} + \frac{\bar{n}^\mu}{2}(m_b - 2E_\gamma). \tag{12.6}$$

Defining our endpoint region by

$$\frac{m_b}{2} - E_\gamma \leq \Lambda_{QCD} \tag{12.7}$$

gives us a mass squared scale of

$$p_X^2 \simeq m_b\Lambda = m_b^2\frac{\Lambda}{m_b} = m_b^2\lambda^2 \tag{12.8}$$

where in the last line we took $\lambda = \sqrt{\frac{\Lambda}{m_b}}$. Taking m_b as Q it is clear that this process is described by SCET$_\text{I}$. Specifically, X will be represented by collinear gluons and quarks while B will be represented by heavy (usoft) quark. Our principal goal is to demonstrate how the effects of momentum scales are factorized in the formula for the photon energy spectrum. To this end we will prove that (12.2) can be factorized as

$$\frac{1}{\Gamma_0}\frac{d\Gamma}{dE_\gamma} = H(m_b,\mu)\int_{2E_\gamma - m_b}^{\Lambda} dk^+\, S(k^+,\mu)J(k^+ + m_b - 2E_\gamma,\mu) \tag{12.9}$$

where $H(m_b,\mu)$ is a calculable quantity arising from hard scale dynamics; $S(k^+,\mu)$ is a non-perturbative soft function; and $J(k^+)$ represents collinear gluons and quarks and is called the jet function.

We begin by matching the QCD current onto SCET to obtain

$$J_\mu = -E_\gamma e^{i(\overline{\mathcal{P}}\frac{n}{2} + \mathcal{P}_\perp - m_b v)\cdot x} C(\overline{\mathcal{P}},\mu)\bar{\xi}_{n,p}W\gamma_\mu^\perp P_L h_v \tag{12.10}$$
$$= -E_\gamma C(m_b,\mu)\bar{\xi}_{n,p}W\gamma_\mu^\perp P_L h_v \tag{12.11}$$

where in the second line we used the label momentum conservation to set $\overline{\mathcal{P}} = m_b$ and $\mathcal{P}_\perp = 0$. Inserting this result into (12.3), we may write

$$\frac{4E_\gamma}{m_b^3} T(E_\gamma) \equiv H(m_b,\mu) T_{\text{eff}}(E_\gamma,\mu) \tag{12.12}$$

where

$$T_{\text{eff}} = i \int d^4x\, e^{i(m_b \frac{\bar{n}}{2} - q)\cdot x} \left\langle \overline{B}_v \right| \mathrm{T} J^\mu_{\text{eff}}(x) J_{\mu\,\text{eff}} \left| \overline{B}_v \right\rangle . \tag{12.13}$$

This gives us a hard amplitude of

$$H(m_b,\mu) = \frac{4E_\gamma^3}{m_b^3} |C(m_b,\mu)|^2. \tag{12.14}$$

Next, we decouple usoft gluons from collinear fields by implementing the standard field redefinitions

$$\xi_{n,p} \to Y \xi^{(0)}_{n,p} \qquad W \to Y\, W^{(0)} Y^\dagger \tag{12.15}$$

thus giving us a new effective current:

$$J^\mu_{\text{eff}} = \overline{\xi}^{(0)}_n W^{(0)} \gamma^\perp_\mu P_L Y^\dagger h_v. \tag{12.16}$$

Substituting this result into (12.13) gives us

$$\begin{aligned} T_{\text{eff}} &= i \int d^4x\, e^{i(m_b \frac{\bar{n}}{2} - q)\cdot x} \left\langle \overline{B}_v \right| \mathrm{T}[\overline{h}_v Y P_R \gamma^\perp_\mu W^{(0)\dagger} \xi^{(0)}_{n,p}](x) [\overline{\xi}^{(0)}_{n,p} W^{(0)} \gamma^\mu_\perp P_L Y^\dagger h_v](0) \left| \overline{B}_v \right\rangle & (12.17) \\ &= -\int d^4x \int \frac{d^4k}{(2\pi)^4} e^{i(m_b \frac{\bar{n}}{2} - q - k)\cdot x} \left\langle \overline{B}_v \right| \mathrm{T}[\overline{h}_v Y](x) P_R \gamma^\perp_\mu \frac{\not{\bar{n}}}{2} \gamma^\mu_\perp P_L [Y^\dagger h_v](0) \left| \overline{B}_v \right\rangle J_P(k) & (12.18) \\ &= \frac{1}{2} \int d^4x \int \frac{d^4k}{(2\pi)^4} e^{i(m_b \frac{\bar{n}}{2} - q - k)\cdot x} \left\langle \overline{B}_v \right| \mathrm{T}[\overline{h}_v Y](x) [Y^\dagger h_v](0) \left| \overline{B}_v \right\rangle J_P(k), & (12.19) \end{aligned}$$

where we defined

$$i \int \frac{d^4k}{(2\pi)^4} \langle 0|\, \mathrm{T}[W^{(0)\dagger} \xi^{(0)}_{n,p}](x) [\overline{\xi}^{(0)}_{n,p} W^{(0)}](0)\, |0\rangle \tag{12.20}$$

with the label P representing the sum of the label momentum carried by the collinear fields. (Additional Derivation)? Now, noting that J_P only depends on the k^+ component of residual momentum k, we may do the k^- and k^+ integrals thus putting x on the light cone

$$\begin{aligned} T_{\text{eff}} &= \frac{1}{2} \int d^4x\, e^{i(m_b \frac{\bar{n}}{2} - q)\cdot x} \delta(x^+) \delta(x_\perp) \int \frac{dk_\perp}{2\pi} e^{-\frac{i}{2} k_+ x_-} \left\langle \overline{B}_v \right| \mathrm{T}[\overline{h}_v Y](x) [Y^\dagger h_v](0) \left| \overline{B}_v \right\rangle J_P(k^+) \\ &= \frac{1}{2} \int dk^+ J_P(k^+) \int \frac{dx^-}{4\pi} e^{-\frac{i}{2}(2E_\gamma - m_b + k^+)x^-} \left\langle \overline{B}_v \right| \mathrm{T}[\overline{h}_v Y] \left(\frac{n}{2} x^-\right) [Y^\dagger h_v](0) \left| \overline{B}_v \right\rangle . & (12.21) \end{aligned}$$

Focusing on the heavy fields, we may then define

$$\begin{aligned}
S(k^+) &\equiv \frac{1}{2}\int \frac{dx^-}{4\pi} e^{-\frac{i}{2}l^+x^-} \langle \overline{B}_v | \mathrm{T}[\overline{h}_v Y](\frac{n}{2}x^-)[Y^\dagger h_v](0) | \overline{B}_v \rangle \quad (12.22)\\
&= \frac{1}{2}\int \frac{dx^-}{4\pi} e^{-\frac{i}{2}l^+x^-} \langle \overline{B}_v | \mathrm{T}e^{x^-\frac{n}{2}\cdot\partial}[\overline{h}_v Y](0)[Y^\dagger h_v](0) | \overline{B}_v \rangle\\
&= \frac{1}{2}\int \frac{dx^-}{4\pi} e^{-\frac{i}{2}l^+x^-} \langle \overline{B}_v | \mathrm{T}[\overline{h}_v Y](0)e^{-x^-\frac{n}{2}\cdot\partial}[Y^\dagger h_v](0) | \overline{B}_v \rangle\\
&= \frac{1}{2}\int \frac{dx^-}{4\pi} e^{-\frac{i}{2}l^+x^-} \langle \overline{B}_v | \mathrm{T}\overline{h}_v Y e^{\frac{ix^-}{2}n\cdot\partial} Y^\dagger h_v | \overline{B}_v \rangle\\
&= \frac{1}{2}\int \frac{dx^-}{4\pi} e^{-\frac{i}{2}l^+x^-} \langle \overline{B}_v | \mathrm{T}\overline{h}_v e^{i\frac{x^-}{2}(in\cdot D_{us})} h_v | \overline{B}_v \rangle\\
&= \frac{1}{2} \langle \overline{B}_v | \overline{h}_v \delta(in\cdot D_{us} - l^\dagger) h_v | \overline{B}_v \rangle . \quad (12.23)
\end{aligned}$$

The Soft function $S(k^+)$ is non-perturbative and encodes information about the usoft dynamics of the B meson. (12.22) shows that we may interpret this result as giving the probability of finding a heavy quark b inside the $\overline{B}$ meson carrying a residual momentum of k^+. Defining $J(k^+) = -\frac{1}{\pi}\mathrm{Im}J_P(k^+)$ and using(12.12),(12.21), (12.22) in (12.2), we have the final result

$$\boxed{\frac{1}{\Gamma_0}\frac{d\Gamma}{dE_\gamma} = \underbrace{H(m_b,\mu)}_{p^2\sim m_b^2 \text{ Hard}} \int_{2E_\gamma - m_b}^{\Lambda} dl^+ \underbrace{S(l^+)}_{p^2\sim\Lambda^2 \text{ Usoft}} \underbrace{J(l^+ + m_b - 2E_\gamma)}_{p^2\sim m_b\Lambda \text{ Collinear}}} \quad (12.24)$$

12.2 Drell-Yan: $pp \to Xl^+l^-$

(**ROUGH)** Our final example will be the Drell-Yan (DY) process $p\bar{p} \to Xl^+l^-$. This is a protype LHC process. The kinematics of this process can be described by the following set of equations.

$$\begin{aligned}
p_A + p_B &= p_X + q \quad (12.25)\\
E_{cm}^2 &= (p_A + p_B)^2 \quad \text{Collision Energy} \quad (12.26)\\
q^2 &\quad : \quad \text{Hard scale of the problem} \quad (12.27)\\
\tau &\equiv q^2/E_{cm}^2 \le 1 \quad (12.28)\\
Y &= \frac{1}{2}\ln\left(\frac{p_b\cdot q}{p_a\cdot q}\right) \quad \text{Total lepton rapidity (angular variable)} \quad (12.29)
\end{aligned}$$

And the analogs of the Bjorken Variables from DIS:

$$x_a \equiv \sqrt{\tau}e^Y, \quad x_b \equiv \sqrt{\tau}e^{-Y}, \quad (12.30)$$

where $\tau \le x_{a,b} \le 1$. We study this process int three distinct energy regions

$$\begin{array}{llll}
\cdot\text{Inclusive:} & \tau\sim 1 & p_x^2 \sim q^2 \sim E_{cm}^2 & x_{a,b}\sim 1,\ \xi_{a,b}\sim 1\\
\cdot\text{Endpoint:} & \tau\to 1 & p_x^2 << q^2 \to E_{cm}^2 & x_{a,b}\to 1,\ \xi_{a,b}\to 1\\
\cdot\text{Isolated:} & \tau\to 0 & p_x^2 >> q^2 & x_{a,b}\to 0,\ \xi_{a,b}\to 0
\end{array} \quad (12.31)$$

We now analyze these specific processes in detail.

Inclusive In this case this process represents an $SCET_I$ problem of hard-collinear factorization. we have a 4-quark operator in SCET, which after a Fierz Identity becomes,

$$[(\bar{\xi}_n W_n)\frac{\bar{n}\!\!\!/}{2}(W_n^\dagger \xi_n)][(\bar{\xi}_{\bar{n}} W_{\bar{n}})\frac{n\!\!\!/}{2}(W_{\bar{n}}^\dagger \xi_{\bar{n}})] \tag{12.32}$$

Remarks:

- $T^A \otimes T^A$ octet structure vanishes under $\langle p_n | \, | \cdot | \, | p_n \rangle$
- When we take $\xi_n \to Y_n \xi_n$ for coupling to soft gluons, the soft wilson lines cancel out.
- This operator encodes information about the PDF because both

$$\langle p_n | \, | \chi_{n,\omega} \frac{\bar{n}\!\!\!/}{2} \chi_{n,\omega} \, | \, | p_n \rangle \quad \text{and} \quad \langle p_{\bar{n}} | \, | \chi_{\bar{n},\omega} \frac{n\!\!\!/}{2} \chi_{\bar{n},\omega} \, | \, | p_{\bar{n}} \rangle \tag{12.33}$$

are defined as PDFs. These PDFs contribute to the differential cross section for this process:

$$\frac{1}{\sigma_0}\frac{d\sigma}{dq^2 dY} = \sum_{i,j} \int_{x_a}^1 \frac{d\xi_a}{\xi_a} \int_{x_b}^1 \frac{d\xi_b}{\xi_b} H_{ij}^{\text{incl}} \left(\frac{x_a}{\xi_a}, \frac{x_b}{\xi_b}, q^2, \mu \right) f_i(\xi_a, \mu) f_j(\xi_b, \mu) \tag{12.34}$$

$$= \left[1 + \mathcal{O}\left(\frac{\Lambda_{QCD}}{\sqrt{q^2}} \right) \right]. \tag{12.35}$$

- As a last important caveat, we not that Glauber Gluons cancel out at leadind order. However, proving this result is out of the scope of our current discussion.

Threshold Limit In the threshold limit only the terms of H_{ij}^{incl} most singular in $1-\tau$ contribute.

$$H_{ij}^{\text{incl}} \quad \to \quad S_{q\bar{q}}^{\text{thr}} \left[\sqrt{q^2}\left(1 - \frac{\tau}{q_a q_b} \right), \mu \right] H_{ij}(q^2, \mu) \left[1 + \mathcal{O}(1-\tau) \right] \tag{12.36}$$

where $i, j = u\bar{u}, \, d\bar{d}, \, \ldots$. The interpretation when we take $\xi_{a,b} \to 1$ is that one parton in each proton carries all the momentum. This is not the dominant LHC region.

Isolated DY The isolated case of DY allows forward jets to carry away part of E_{cm}, so $\xi_{a,b} \to 1$. It also restricts thr central region to still only have soft radiation (the signal region is background free). To guarantee this requires an experimental observation. **Observable:** $p_X = B_a + B_b$. There are two hemispheres perpendicular to the beam axis.

$$B_a^+ = n_a \cdot B_a = \sum_{k \epsilon a} n_a \cdot p_k \tag{12.37}$$

$$= \sum_{k \epsilon a} E_k (1 + \tanh Y_k) e^{-2Y_k} \tag{12.38}$$

We expect the plus momenta for n- collinear radiation to be small. We find that this is indeed the case becuase

$$B_a^+ \leq Q e^{-2Y\omega t} << Q \tag{12.39}$$

and there is an identical expression for B_b^+. For the n-collinear proton (a) and jet (a), we do not merely get a PDF from the hard-collinear-soft factorization. We get something new called a beam function. The

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