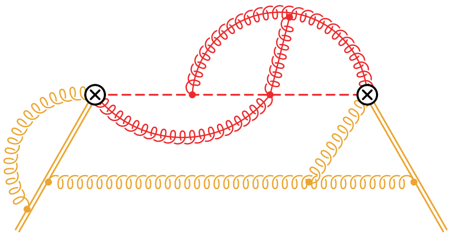


Here the jet mass is also the mass of the hadronic final state, and the situation which dominates the phenomenology has $m_X^2 \sim Q\Lambda_{\rm QCD}$. We have collinear modes for the jet, and ultrasoft modes with $p_{us}^2 \sim \Lambda_{\rm QCD}^2$ which are the constituents of the B meson for this inclusive decay. Often the region where $m_X^2 \ll Q^2$ is known at the endpoint region since $E \sim m_B/2 - \Lambda_{\rm QCD}$ and hence is close to the physical endpoint $E = m_B/2$. (The case $m_X^2 \sim Q^2$ is then known as the local OPE region where the traditional HQET operator product expansion analysis suffices.) The picture of the modes for this case are shown in Fig. 3, and indeed yield an example of an SCET$_{\rm I}$ theory with only one collinear mode.

3 Ingredients for SCET

Our objective in this section is to expand QCD and formulate collinear and ultrasoft degrees of freedom. In doing so, we will derive power counting expressions for operators and see what form the quark Lagrangian takes in a SCET theory.

3.1 Collinear Spinors

We begin our exploration by considering the decomposition in the collinear limit of Dirac spinors $u(p)$ for particles and $v(p)$ for antiparticles. We will derive the collinear spinors by considering the expansion in momentum components, but then will convert this result into a decomposition into two types of terms rather than an infinite expansion.

For a collinear momentum $p^\mu = (p^0, p^1, p^2, p^3)$ we have $p^- = p^0 + p^3 \gg p_\perp^{1,2} \gg p^+ = p^0 - p^3$ so

$$\frac{\vec{\sigma}\cdot\vec{p}}{p^0} = \sigma_3 + \ldots\,, \tag{3.1}$$

where the terms in the $+\ldots$ are smaller. Keeping only the leading term gives us the spinors

$$\begin{aligned} u(p) &= \frac{(2p^0)^{1/2}}{\sqrt{2}} \begin{pmatrix} \mathcal{U} \\ \frac{\vec{\sigma}\cdot\vec{p}}{p^0}\mathcal{U} \end{pmatrix} \quad \Longrightarrow \quad u_n = \sqrt{\frac{p^-}{2}} \begin{pmatrix} \mathcal{U} \\ \sigma^3\mathcal{U} \end{pmatrix} \\ v(p) &= \frac{(2p^0)^{1/2}}{\sqrt{2}} \begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{p^0}\mathcal{V} \\ \mathcal{V} \end{pmatrix} \quad \Longrightarrow \quad v_n = \sqrt{\frac{p^-}{2}} \begin{pmatrix} \sigma^3\mathcal{V} \\ \mathcal{V} \end{pmatrix} \end{aligned} \tag{3.2}$$

where here $\mathcal{U}$ and $\mathcal{V}$ are each either $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. From this analysis we see that in the collinear limit both quark and antiquarks remain as relevant degrees of freedom (and indeed, there is no suppression for pair creation from splitting). We also see that both spin components remain in each of the spinors. Recalling our default definitions of n^μ and $\bar{n}^\mu$, we can calculate their contractions with the gamma matrix,

$$\not{n} = \gamma_0 - \gamma_3 = \begin{pmatrix} \mathbb{1} & -\sigma^3 \\ \sigma^3 & -\mathbb{1} \end{pmatrix}, \qquad \bar{\not{n}} = \gamma_0 + \gamma_3 = \begin{pmatrix} \mathbb{1} & \sigma^3 \\ -\sigma^3 & -\mathbb{1} \end{pmatrix}. \tag{3.3}$$

Multiplying the first matrix by u_n or v_n from (3.2) gives the following relations

$$\not{n} u_n = 0\,, \qquad \not{n} v_n = 0\,. \tag{3.4}$$

These can be recognized as the leading term in the equations of motion $\not{p}u(p) = \not{p}v(p) = 0$ when expanded in the collinear limit. We can also define projection operators

$$P_n = \frac{\not{n}\not{\bar{n}}}{4} = \frac{1}{2}\begin{pmatrix} \mathbb{1} & \sigma^3 \\ \sigma^3 & \mathbb{1} \end{pmatrix}, \qquad P_{\bar{n}} = \frac{\not{\bar{n}}\not{n}}{4} = \frac{1}{2}\begin{pmatrix} \mathbb{1} & -\sigma^3 \\ -\sigma^3 & \mathbb{1} \end{pmatrix}, \tag{3.5}$$

and then we have the relations

$$P_n u_n = \frac{\not{n}\not{\bar{n}}}{4} u_n = u_n, \qquad P_n v_n = \frac{\not{n}\not{\bar{n}}}{4} v_n = v_n. \tag{3.6}$$

The bottom line of this expansion is that when a hard interaction produces a collinear fermion or antifermion it will be the components obeying the spin relations in Eqs. (3.4) and (3.6) that appear at leading order.

For later purposes it will be useful to decompose the QCD Dirac field ψ into a field ξ_n that obeys these spin relations. From $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ we note that

$$\frac{\not{n}\not{\bar{n}}}{4} + \frac{\not{\bar{n}}\not{n}}{4} = \mathbb{1}, \tag{3.7}$$

which allows us to write ψ in terms of two fields,

$$\psi = P_n\psi + P_{\bar{n}}\psi = \hat{\xi}_n + \varphi_{\bar{n}} \tag{3.8}$$

where we defined

$$\hat{\xi}_n = P_n\psi = \frac{\not{n}\not{\bar{n}}}{4}\psi\,, \qquad \varphi_{\bar{n}} = P_{\bar{n}}\psi = \frac{\not{\bar{n}}\not{n}}{4}\psi. \tag{3.9}$$

These fields satisfy the desired spin relations

$$\not{n}\xi_n = 0\,, \qquad P_n\xi_n = \xi_n\,, \qquad \not{\bar{n}}\varphi_{\bar{n}} = 0\,, \qquad P_{\bar{n}}\varphi_{\bar{n}} = \varphi_{\bar{n}}\,. \tag{3.10}$$

The label n on $\hat{\xi}_n$ reminds us that it obeys these relations and that we will eventually be expanding about the n-collinear direction. Note that here we denote the collinear field components with a hat, as in $\hat{\xi}_n(x)$, since there are still further manipulations that are required before we arrive at our final SCET collinear field $\xi_n(x)$. Nevertheless both $\hat{\xi}_n$ and ξ_n satisfy these spinor relations.

Having defined $\hat{\xi}_n = P_n\psi$, the corresponding result for the spinors is $u_n = P_n u(p)$ and $v_n = P_n v(p)$, which do not precisely reproduce the lowest order expanded results in Eq. (3.2). Instead we find

$$\begin{aligned} u_n &= \frac{1}{2}\begin{pmatrix} \mathbb{1} & \sigma_3 \\ \sigma_3 & \mathbb{1} \end{pmatrix} \sqrt{p^0} \begin{pmatrix} \mathcal{U} \\ \frac{\vec{\sigma}\cdot\vec{p}}{p_0}\mathcal{U} \end{pmatrix} = \frac{\sqrt{p^0}}{2} \begin{pmatrix} \left(1 + \frac{p_3}{p_0} - \frac{(i\vec{\sigma}\times\vec{p}_\perp)_3}{p^0}\right)\mathcal{U} \\ \sigma_3\left(1 + \frac{p_3}{p_0} - \frac{(i\vec{\sigma}\times\vec{p}_\perp)_3}{p^0}\right)\mathcal{U} \end{pmatrix} \\ &= \sqrt{\frac{p^-}{2}} \begin{pmatrix} \tilde{\mathcal{U}} \\ \sigma_3\tilde{\mathcal{U}} \end{pmatrix} \end{aligned} \tag{3.11}$$

where the two component spinor is

$$\tilde{\mathcal{U}} = \sqrt{\frac{p^0}{2p^-}} \left(1 + \frac{p_3}{p_0} - \frac{(i\vec{\sigma}\times\vec{p}_\perp)_3}{p^0}\right)\mathcal{U}\,. \tag{3.12}$$

The same derivation gives

$$v_n = \sqrt{\frac{p^-}{2}} \begin{pmatrix} \sigma_3 \tilde{\mathcal{V}} \\ \tilde{\mathcal{V}} \end{pmatrix} \tag{3.13}$$

where $\tilde{\mathcal{V}}$ is defined in terms of $\mathcal{V}$ by a formula analogous to Eq. (3.12). Since the spin relations in Eqs. (3.4) and (3.6) do not depend on the form of the two component spinors ($\tilde{\mathcal{U}}$ versus $\mathcal{U}$ etc), they remain true. We will see later that the results for the u_n and v_n spinors involving $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{V}}$ rather than $\mathcal{U}$ and $\mathcal{V}$ are required to avoid breaking a reparameterization symmetry in SCET. The extra terms appearing in the definition of $\tilde{\mathcal{U}}$ ensure the proper structure under reparameterizations of the lightcone basis. Finally we note that

$$\sum_s \tilde{\mathcal{U}}^s \tilde{\mathcal{U}}^{\dagger\, s} = \mathbb{1}_{2\times 2} \tag{3.14}$$

Thus if we take the product of u_n spinors

$$u_n \bar{u}_n = \frac{p^-}{2} \begin{pmatrix} \tilde{\mathcal{U}}\tilde{\mathcal{U}}^\dagger & -\tilde{\mathcal{U}}\tilde{\mathcal{U}}^\dagger \sigma_3 \\ \sigma_3 \tilde{\mathcal{U}}\tilde{\mathcal{U}}^\dagger & -\sigma_3 \tilde{\mathcal{U}}\tilde{\mathcal{U}}^\dagger \sigma_3 \end{pmatrix}, \tag{3.15}$$

and sum over spins, we have

$$\sum_s u_n^s \bar{u}_n^s = \frac{\not{n}}{2}\, \bar{n}\cdot p\,. \tag{3.16}$$

For later convenience we write down a set of projection operator identities easily derived from $n^2 = 0$, $\bar{n}\cdot n = 2$, and/or hermitian conjugation $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$:

$$P_n P_{\bar{n}} = 0\,, \qquad P_n P_n = P_n\,, \qquad P_n \not{\bar{n}} = P_{\bar{n}} \not{n} = 0\,, \qquad P_n \not{n} = \not{n}\,, \qquad P_{\bar{n}} \not{\bar{n}} = \not{\bar{n}}\,, \qquad P_n^\dagger = \gamma_0 P_{\bar{n}} \gamma_0\,. \tag{3.17}$$

None of these results depends on making the canonical back-to-back choice for $\bar{n}$. The last result is useful for the computation of $\bar{\hat{\xi}}_n$ from $\hat{\xi}_n = P_n \psi$, i.e.

$$\bar{\hat{\xi}}_n = \hat{\xi}_n^\dagger \gamma^0 = \psi^\dagger P_n^\dagger \gamma^0 = \overline{\psi}\, P_{\bar{n}}\,. \tag{3.18}$$

Thus just like the relations for $\hat{\xi}_n$ or ξ_n we have the following relations for $\bar{\hat{\xi}}_n$ or $\bar{\xi}_n$:

$$\bar{\xi}_n \not{n} = 0\,, \qquad \bar{\xi}_n P_n = 0\,, \qquad \bar{\xi}_n P_{\bar{n}} = \bar{\xi}_n \frac{\not{\bar{n}}\not{n}}{4} = \bar{\xi}_n\,. \tag{3.19}$$

In addition to our collinear decomposition of the Dirac spinors and field, we will also need spinors and quark fields for the ultrasoft degrees of freedom. However, since all ultrasoft momenta are homogeneous of order λ^2 and the scaling of momenta does not affect the corresponding components of the ultrasoft spinors, which are the same as those in QCD.

3.2 Collinear Fermion Propagator and ξ_n Power Counting

Having considered the decomposition of spinors in the collinear limit, we now turn to the fermion propagator in the collinear limit. Here $p^2 + i0 = \bar{n}\cdot p\, n\cdot p + p_\perp^2$, and since both of these terms are $\sim \lambda^2$ there is no expansion of the denominator of the propagator. We can however expand the numerator by keeping only the large $\bar{n}\cdot p$ momentum, as

$$\frac{i\not{p}}{p^2+i0} = \frac{i\not{n}}{2}\frac{\bar{n}\cdot p}{p^2+i0} + \ldots = \frac{i\not{n}}{2}\frac{1}{n\cdot p + \frac{p_\perp^2}{\bar{n}\cdot p} + i0\,\mathrm{sign}(\bar{n}\cdot p)} + \ldots \tag{3.20}$$

The fermion-gluon coupling will be proportional to $\bar{\not{n}}/2$ and hence will form a projector P_n when combined with the $\not{n}/2$ from the propagator. Therefore the displayed term in the propagator has overlap with our spinors u_n and v_n, just giving $P_n u_n = u_n$ etc. The fact that both $+i0$ and $-i0$ occur in the expanded propagator is a reflection of the fact that the lowest order SCET Lagrangian will contains both propagating particles ($\bar{n}\cdot p > 0$) and propagating antiparticles ($\bar{n}\cdot p < 0$).

The leading collinear propagator displayed in Eq. (3.20) should be obtained from a time-ordered product of the effective theory field, $\langle 0|T\hat{\xi}_n(x)\bar{\hat{\xi}}_n(0)|0\rangle$. At this point we can already identify the λ power counting for the field $\hat{\xi}_n$ by noting that if its propagator has the form in Eq. (3.20) then its action must be of the form

$$L_n^{(0)} = \int d^4x\, \mathcal{L}_n^{(0)} = \int \underbrace{d^4x}_{\mathcal{O}(\lambda^{-4})}\ \underbrace{\bar{\hat{\xi}}_n}_{\mathcal{O}(\lambda^a)}\ \frac{\bar{\not{n}}}{2}\big[\underbrace{in\cdot\partial + \ldots}_{\mathcal{O}(\lambda^2)}\big]\ \underbrace{\hat{\xi}_n}_{\mathcal{O}(\lambda^a)} \sim \lambda^{2a-2}\,. \tag{3.21}$$

Here we used the fact that $d^4x = \frac{1}{2}(dx^+)(dx^-)(d^2x_\perp) \sim (\lambda^0)(\lambda^{-2})(\lambda^{-1})^2 \sim \lambda^{-4}$ where the scaling for the coordinates x^μ follows from those for the collinear momenta by writing $x\cdot p_c = x^+p_c^- + x^-p_c^+ + 2x_\perp\cdot p_c^\perp$ and demanding that the terms in this sum are all $\mathcal{O}(1)$. In (3.21) we assigned $\hat{\xi}_n \sim \lambda^a$ with the goal of determining the value of a. To do this we take the standard approach of assigning a power counting to the leading order kinetic term in the action so that $L_n^{(0)} \sim \lambda^0$, which gives

$$\hat{\xi}_n \sim \xi_n \sim \lambda\,. \tag{3.22}$$

Even though we have not fully considered all the issues needed to define the SCET collinear field ξ_n, the further manipulations we will make in section 4 below will not effect its power counting, so we have also recorded here the fact that the SCET field $\xi_n \sim \lambda$. Note that this scaling dimension does not agree with the collinear quark fields mass dimension since $[\hat{\xi}_n] = [\xi_n] = 3/2$. This is simply a reflection of the fact that the SCET power counting for operators is not a power counting in mass dimensions. The observant reader will notice that the λ scaling of the collinear field is the same as its twist, and indeed the SCET power counting reduces to a (dynamic) twist expansion when the latter exists.

3.3 Power Counting for Collinear Gluons and Ultrasoft Fields

Similar to our procedure for the collinear fermion field, we can analyze the collinear gluon field A_n^μ in our n-collinear basis to determine the λ scaling of its components. This information is necessary to formulate the importance of operators in SCET. We begin by writing the full theory covariant gauge gluon propagator, but we label the fields as $A_n^\mu(x)$ to denote the fact that we will be considering a n-collinear momenta:

$$\int d^4x\, e^{ik\cdot x}\,\langle 0|\,\mathrm{T}A_n^\mu(x)A_n^\nu(0)\,|0\rangle = -\frac{i}{k^2}\left(g^{\mu\nu} - \tau\frac{k^\mu k^\nu}{k^2}\right) = -\frac{i}{k^4}\left(k^2 g^{\mu\nu} - \tau\, k^\mu k^\nu\right), \tag{3.23}$$

where τ is our covariant gauge fixing parameter. From our standard power counting result from the light-cone coordinate section, we know that $k^2 = k_+k_- + k_\perp^2 = Q^2\lambda^2$. So the $1/k^4$ on the RHS matches up with the scaling of the collinear integration measure

$$d^4x \sim \lambda^{-4} \sim \frac{1}{(k^2)^2} \tag{3.24}$$

Thus the quantity in the final parentheses in (3.23) must be the same order as the product of $A_n^\mu(x)A_n^\nu(0)$ fields. If both of the $\mu\nu$ indices are $\perp$ then both of the terms in these parantheses are $\sim \lambda^2$, so therefore we must have $A^\mu_{n\perp} \sim \lambda$. If one index is $+$ and the other $-$ then again both terms are the same size and we find $A_n^+ A_n^- \sim \lambda^2$. To break the degeneracy we take both indices to be $+$, then $g^{++} = 0$, $(n\cdot k)^2 \sim \lambda^4$, so $A_n^+ \sim \lambda^2$ and $A_n^- \sim \lambda^0$. Other combinations also lead to this result, namely that the components of the collinear gluon field scales in the same way as the components of the collinear momentum

$$A_n^\mu \sim k^\mu \sim (\lambda^2, 1, \lambda). \tag{3.25}$$

This result is not so surprising considering that if we are going to formulate a collinear covariant derivative $D^\mu = \partial^\mu + igA^\mu$ with collinear momenta ∂^μ and gauge fields, then for each component both terms must have the same λ scaling. Indeed imposing this property of the covariant derivative is another way to derive Eq. (3.25).

The same logic can be used to derive the power counting for ultrasoft quark and gluon fields. Since the momentum $k^\mu_{us} \sim (\lambda^2, \lambda^2, \lambda^2)$ the measure on ultrasoft fields scales as $d^4x \sim \lambda^{-8}$. Also the result is now uniform for the components of A^μ_{us}. Once again we find that the gluon field scales like its momentum. For the ultrasoft quark we have the Lagrangian $\mathcal{L} = \bar\psi_{us} i\not{D}_{us}\psi_{us}$ with $iD^\mu_{us} = i\partial^\mu + gA^\mu_{us} \sim \lambda^2$. Therefore $\bar\psi_{us}\psi_{us} \sim \lambda^6$. All together we have

$$A^\mu_{us} \sim (\lambda^2, \lambda^2, \lambda^2)\,, \qquad\qquad \psi_{us} \sim \lambda^3\,. \tag{3.26}$$

For a heavy quark field that is ultrasoft the Lagrangian is $\mathcal{L}_{\rm HQET} = \bar h_v^{us} iv\cdot D_{us} h_v^{us}$ which is again linear in the derivative, so $h_v^{us} \sim \lambda^3$ as well.

For completeness we also remark that the power counting for momenta determines the power counting for states. For one-particle states of collinear particles (with a standard relativistic normalization):

$$\langle p'|p\rangle = 2p^0\delta^3(\vec p - \vec p\,') = p^-\delta(p^- - p'^-)\delta^2(\vec p^\perp - \vec p\,') \sim \lambda^{-2} \tag{3.27}$$

Thus the single particle collinear state has $|p\rangle \sim \lambda^{-1}$ for both quarks and gluons. Given the scaling of the collinear quark and gluon fields, this implies power counting results for the polarization objects. The collinear spinors $u_n \sim \xi_n|p\rangle \sim \lambda^0$ which is consistent with our earlier Eq. (3.11). For the physical $\perp$ components of polarization vectors for collinear gluons we also find $\epsilon^\mu_\perp \sim \lambda^0$.

Of particular importance in the result in Eq.(3.25) is the fact that $\bar n\cdot A_n = A_n^- \sim \lambda^0$, indicating that there is no λ supression to adding A_n^- fields in SCET operators. To understand the relevance of this result we consider in the next section an example of matching for an external current from QCD onto SCET.

3.4 Collinear Wilson Line, a first look

To see what impact there is to having a set of gauge fields $\bar n\cdot A_n \sim \lambda^0$ lets consider as an example the process $b \to ue\bar\nu$, where the b quark is heavy and decays to an energetic collinear u quark. This process has the advantage of only invoving a single collinear direction. This decay has the following weak current with QCD fields

$$J_{QCD} = \bar u\,\Gamma b \tag{3.28}$$

where $\Gamma = \gamma^\mu(1-\gamma^5)$. Without gluons we can match this QCD current onto a leading order current in SCET by considering the heavy b field to be the HQET field h_v and the lighter u field by the SCET field ξ_n. This is shown in Fig. 4 part (a), where we use a dashed line for collinear quarks. The resulting SCET operator is

$$\bar\xi_n\Gamma h_v. \tag{3.29}$$

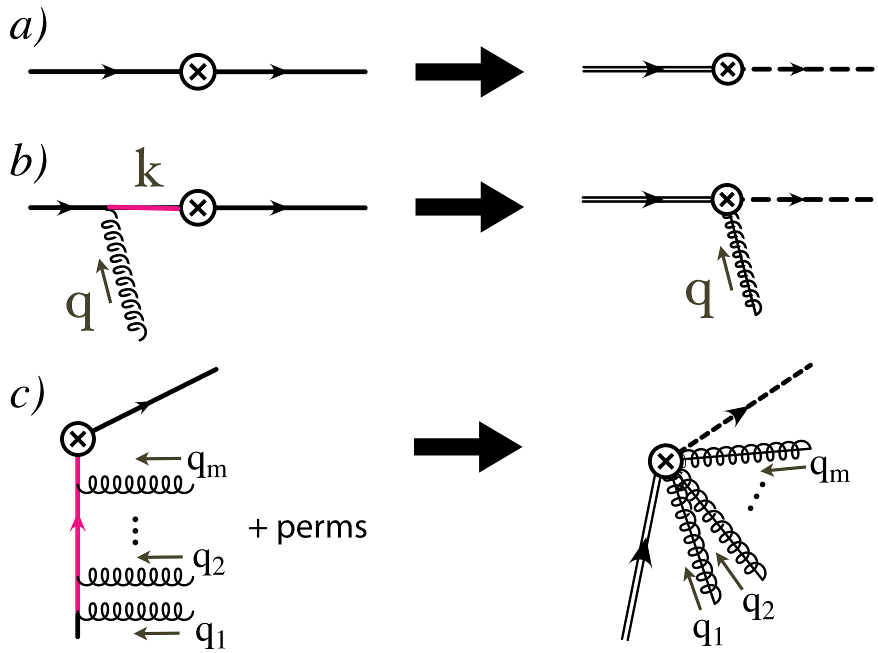


Figure 4: Tree level graphs for matching the heavy-to-light current.

Next we consider the case where an extra A_n^- gluon is attached to the heavy quark. This process is shown in Fig.4 part (b) and leads to an offshell propagator, shown by the pink line, that must be integrated out when constructing the EFT. The full theory amplitude for this process is (replacing external spinors and polarization vectors by SCET fields):

$$\begin{aligned} A_n^{\mu\,A}\,\bar{\xi}_n\Gamma\,\frac{i(\not{k}+m_b)}{k^2-m_b^2}\,igT^A\gamma_\mu h_v &= -g\Big(\frac{n^\mu}{2}\bar{n}\cdot A_n^A\Big)\,\bar{\xi}_n\Gamma\,\frac{[m_b(1+\not{v})+\not{q}]}{2m_b v\cdot q+q^2}T^A\gamma_\mu h_v \\ &= -g\bar{n}\cdot A_n^A\,\bar{\xi}_n\Gamma\left[\frac{m_b(1+\not{v})+\frac{\not{n}}{2}\bar{n}\cdot q}{m_b v\cdot n\,\bar{n}\cdot q}+\dots\right]T^A\frac{\not{n}}{2}h_v \\ &= -g\bar{n}\cdot A_n^A\,\bar{\xi}_n\Gamma\left[\frac{\frac{\not{n}}{2}(1-\not{v})+v\cdot n}{v\cdot n\,\bar{n}\cdot q}+\dots\right]T^A h_v \\ &= \bar{\xi}_n\Big(\frac{-g\,\bar{n}\cdot A_n}{\bar{n}\cdot q}\Big)\Gamma h_v \end{aligned} \tag{3.30}$$

In the first equality we have used the fact that the incoming b quark carries momentum $m_b v^\mu$, that $k = m_b v + q$ so that $k^2 - m_b^2 = 2m_b v\cdot q + q^2$, and that

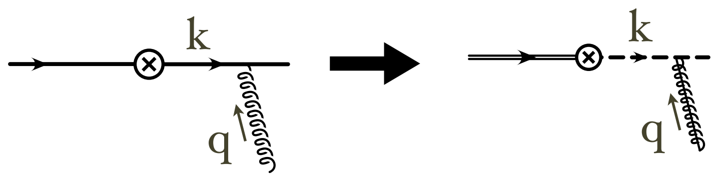
$$A_n^\mu = \frac{n^\mu}{2}\underbrace{\bar{n}\cdot A_n}_{O(\lambda^0)} + \frac{\bar{n}^\mu}{2}\underbrace{n\cdot A_n}_{O(\lambda^2)} + \underbrace{A_\perp^\mu}_{O(\lambda)} \tag{3.31}$$

where we can keep only the $\sim\lambda^0$ term. In the second equality in Eq. (3.30) we have expanded the numerator and denominator of the propagator in λ and kept only the lowest order terms. Since $m_b v\cdot n\,\bar{n}\cdot q \sim Q^2\lambda^0$ we see that the propagator is offshell by an amount of $\sim Q^2$, and hence is a hard propagator that we must integrate out when constructing the corresponding SCET operator. In the third equality we use $\bar{n}\!\!\!/^2 = 0$ and pushed the $\bar{n}\!\!\!/$ through to the left. Noting that $(1 - v\!\!\!/)h_v = 0$, the fourth equality gives the final leading order result from this calculation. Thus we we see that in SCET integrating out offshell hard propagators that are induced by $\bar{n} \cdot A_n$ gluons leads to an operator for the leading order current with one collinear gluon coming out of the vertex, pictured on the RHS of Fig. 4 part (b).

Inspecting the final result in Eq. (3.30) we see that, in addition to being a great simplification of the original QCD amplitude for this gluon attachments, it is indeed of the same order in λ as the result in Eq. (3.29). Indeed it straightforward to prove that the same $(-g\bar{n} \cdot A_n/\bar{n} \cdot q)$ result will be obtained if we replace the heavy quark by a particle that is not n-collinear, such as a collinear quark in a different direction n' where $n \cdot n' \gg \lambda^2$. The sum of collinear momenta in the n and n' directions will also be offshell, for example when we add two back-to-back collinear momenta $(p_n + p_{\bar{n}})^2 \sim \lambda^0$. In all these situations we find operators with additional $\bar{n} \cdot A_n \sim \lambda^0$ fields.

In summary, the off-shell quark has been integrated out and its effects have been parameterized by an effective operator. This was necessary because the virtual quark resulting from the interaction of a heavy quark or a n' collinear particle with a n-collinear gluon yields an off-shell momentum.

This result can be contrasted with what happens if we attach a single $\bar{n} \cdot A_n$ collinear gluon field to the light collinear u quark, as shown below:



Calling the final u quarks momentum p we have $k^\mu = p^\mu - q^\mu$. However here since both p and q are n-collinear the propagator momentum k^μ also has n-collinear scaling. In particular $k^2 \sim \lambda^2$ and is not offshell, it instead represents a propagating mode within the effective theory. Thus this interaction is reproduced in SCET by a collinear propagator followed by a leading order Feynman rule that couples the $\bar{n} \cdot A_n$ field to the collinear quark. Thus this diagram corresponds to a time ordered product of the leading order SCET current $J^{(0)}$ with the leading order Lagrangian $\mathcal{L}_n^{(0)}$. If we attach more collinear gluons to the light u quark, the same remains true. We never get an offshell propagator that we have to integrate out when we have an interaction between n-collinear particles. Indeed we will also find that the components $n \cdot A_n$ and $A_n^\perp$ couple at leading order in T-products like the one shown above, so there is nothing special about the $\bar{n} \cdot A_n$ components for these diagrams.

Lets now consider the situation of multiple gluon emission from the heavy quark. In this case we again have offshell propagators, which are represented by the pink line in Fig. 4 part (c). By inspection, it is clear that the generalization from one gluon emission to k gluon emissions with momenta $q_1, \ldots, q_k$ and propagators with momenta $q_1, q_1 + q_2, \ldots, \sum_{i=1}^k q_i$ yields

$$\bar{\xi}_n \sum_{\text{perm}} \frac{(-g)^k}{k!} \left(\frac{\bar{n} \cdot A_{q_1} \cdots \bar{n} \cdot A_{q_k}}{[\bar{n} \cdot q_1][\bar{n} \cdot (q_1 + q_2)] \cdots [\bar{n} \cdot \sum_{i=1}^k q_i]} \right) \Gamma h_v \tag{3.32}$$

Here the sum of permutations (perms) of the $\{q_1, \ldots, q_k\}$ momenta accounts for the fact that we must consider diagrams with crossed gluon lines on the LHS of Fig. 4 part (c). We also include the factor of $k!$ as a symmetry factor to account for the fact that all k gluon fields are localized and identical and may be contracted with any external gluon state. Finally, by summing over the number of possible gluon emissions, we can write the complete tree level matching of the QCD current to the SCET current,

$$J_{\text{SCET}} = \bar{\xi}_n W_n \Gamma\, h_v\,, \tag{3.33}$$

8.851 Effective Field Theory
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