MITOCW | 2. Dimensional Power Counting

IAIN STEWART: So last time, we were talking about the standard model as an effective field theory. And we decided that the power counting would be this, epsilon, the masses of the particles in the standard model, the scales in the standard model divided by some new physics scale, scale outside the standard model. And I made the statement that this was connected to operator dimension, but I didn't make that precise.

And I want to do that now as the first thing we do today.

So let's spend a few minutes and talk about marginal, irrelevant, and relevant operators and their connection to power counting.

I'm going to write power counting over and over again. And I'm going to abbreviate it p.c. from now on. So let's consider an effective field theory.

It'll be a scalar effective field theory in d dimensions, standard kinetic term, mass term, phi 4 term. So it'll be a phi 4 scalar field theory.

It will be an effective theory, so we won't stop there. And I'll just write down up to phi 6. And then, in principle, I could keep going.

So we can look at the dimensions of the various objects here. The action with our units is dimensionless. h bar and c are 1.

So the mass dimensions of the field in d dimensions are d minus 2 over 2 since the dimensions of ddx or minus d. We have to compensate for that. And we have to compensate for the two derivatives.

The canonically normalized kinetic term tells us what the dimensions of phi are. And then we can work out the dimensions of everything else, so mass squared dimension 2. Tau dimension-- 6 minus 2d. Lambda could be dimension 0 if d is 4. OK. So hopefully-- somewhat familiar stuff. So let's say we want to study a correlation function of a bunch of phis at different space time points.

And we want to look at it at long distance. Long distance is small momenta. So the way I'm going to make the distance long is I'm going to say that all these x's that are appearing in my phis, x1 through xn, I'm going to redefine them as some s common parameter times x prime.

And then I'm just going to take s goes to infinity with the x prime fixed. So that makes all the x's large. So when I do that, if I make a redefinition like that, I can mess up the normalization of my kinetic term.

It'll no longer be canonical, but I can fix that by just redefining my field. And the way to do that is to do the following. To find a new, phi prime, it's equal to the old field, but rescaled by an s.

And the outcome of that is that we get an action for the phi prime field written in terms of prime coordinates, which has a kinetic term that's the same form. But then s's start showing up in the other places.

And if you look at the powers of the s's that are showing up, it's related also to the powers of these parameters. Yeah.

AUDIENCE: Are you sure about the powers of lambda tau, for the dimension lambda tau?

IAIN STEWART: Did I get it backwards? Should it be d minus 4?

AUDIENCE: I think it should be d minus 4. And then the other one should be d minus 6 [INAUDIBLE].

IAIN STEWART: Yeah. That looks right. Oh, you have to be-- so let's see. There's d's here, right? So it's not d.

AUDIENCE: Oh.

IAIN STEWART: You've got to keep--

AUDIENCE: [INAUDIBLE]

IAIN STEWART: Yeah. I stick by what I wrote. Check it. All right, so let's look at the correlation function in terms of the phi prime because the phi prime is just a function of x primes. And the x primes are holding fixed.

So if we rescale everything in terms of the x primes, then we have some matrix element that's not growing with s. We can make all the s's explicit. So we take our original guy, which in terms of our new variables looks like that.

We make this redefinition. We get some powers of s out front. And then we get something which is just in terms of the x prime and won't grow with s.

So we can study this in various dimensions if we wanted to. Let's, for simplicity and also since it's the most common case we're interested in, take d equals 4. I still may write d's, but let's from here on take d equals 4 and ask the question, what happens is s gets large?

So now, we've made all the s's explicit. This is something that you often do when you're doing effective field theory. You can figure out how you're going to study the large distance behavior.

You want to make the parameter that's controlling that limit explicit, so you can see it. So it's not hiding anywhere. And that's what we've done with this algebra.

So as s goes to infinity, because we have this explicit s squared there, the m squared term is becoming more and more important. It's called relevant.

Child term is becoming less important. Because if I put in d equals 4, then this is s to the minus 2. It's tracking the s's, making it less important as s grows. And the lambda term is equally as important as it was before.

And the terminology that goes along with this is an association. So that was a statement about parameters. We could also make a statement about operators since obviously they were part of the story here that gave the s factors.

So we would say that phi squared is a relevant operator. The phi 4 is marginal. And phi 6 is irrelevant.

And you can see, because of the argument that we made, that this was just directly connected to dimension, so either to the dimension of the operators or to the dimension of the parameters.

OK, so we're connecting something that we can say, which is the power counting. In this case, we're controlling that with s using s as our control parameters to look at long distances. And we're seeing that that gets connected to dimensions of operators.

Is there any questions? So let's take s finite, but large. Usually, we're not interested in taking it all the way to infinity. Although we may make it as large as we want to study some long distance behavior.

And so what I just said is that we can see, from the powers of s, the importance of the various terms. Relevant terms are more important than marginal terms. And marginal terms are more important than irrelevant terms. The words say it all.

So that means that, if you want to think of how to do the power counting and you don't want to think of introducing this s, since that was kind of just our choice-- we introduced it as a way of thinking about this question. But if we went back to the original action, we should have a way of doing the power counting from that without having to do this rescaling. And we know how to do that now.

This exercise teaches us that we can just look at mass dimensions of the parameters to do the power counting. So if we just associate a power to the parameters, we're still in d equals 4.

Then we would get this association, this being the statement that it's relevant, marginal, and irrelevant. And we can do a power counting in this lambda nu.

And we can then say, in a language which would be familiar from Feynman diagrams where we do everything in momentum space, that the momentum we want to study, p, has to be much less than this lambda nu. And we'll do the power counting to lambda nu. And that will make the, for example, tau term an irrelevant, less important, operator.

So there's one comment here. We did the scalar field theory just because it's simplest. It also has a relevant operator. And we see that relevant operators actually can be dangerous because we'd like to set the power counting for the whole problem by the kinetic term.

We'd like to say that the kinetic term, which was canonically normalized and had no s's in it, we'd like to say that that was relevant, that that's part of the leading order Lagrangian. But when we went through it, we found something that was more relevant, the master. Phi squared could become even larger than the kinetic term.

So we have to be careful about relevant operators. And then this is, of course, related to the Higgs fine tuning. So even though I'm using a scalar field theory, I'm, for the most part, going to just fine tune and ignore this problem, since if I was using something fermionic field theory, I could set things up so I could ignore it from the start. But still using a scalar field theory is convenient.

So I want to also come back to something else that we mentioned last time and go into a little more detail. And that is the discussion of divergences. So last time, we said that there was two different ways of thinking about renormalizability, a traditional sense of renormalizability, renormalizability of the standard model, or an effective field theory way of thinking about renormalizability.

So I want to come back to that with our example of this scalar field theory. So let's get rid of this issue of having something that can upset the power counting either by taking m to be 0 or just fine tuning it to be small. And what that means is I just demand that, as s grows, I shrink m.

And if I do that, then I can, by hand, tune term and this term to be the same size. So if you like, I'm assigning a scaling to m in order to make the mass term be always as important as the kinetic term. So with that little proviso, we can start thinking about divergences. And when we start drawing Feynman diagrams, they will generically have divergences.

So we could have two four-point interactions, which I label by lambda, because that's the parameter that shows up in the Feynman rule. If this is k and this is some k plus p, then this guy is going to have two pairs of lambda. And it's going to be some integral like that.

We won't worry about overall factors here. I'm regulating with dimensional regularization. I'll often do that when it's convenient for us. If you ask how this integral diverges, you could ask how it diverges just in terms of thinking about it in terms of some parameter that's controlling the ultraviolet, like a cutoff.

So even if I am using dim reg, I could ask, what's the power of the divergence? And it diverges as, in d dimensions, lambda to d minus 4. You say d minus 4 is the degree of the divergence. And that's because you have d powers of k from the measure and minus 4 from the propagators.

So if d is equal to 4, you say degree of divergence is 0, but that means log diversion. So if you take d equals 4 in a UV, 4 powers of k downstairs, d upstairs-- but if d is 4, that's 4 upstairs and 4 downstairs.

So it's scaling length dk over k. So I made it Euclidean. That's exactly what it would become.

And that's like a log. So it's a log of the cut off. So it's a 1 over epsilon in dim reg where d is 4 minus 2 epsilon.

And if you just want to think about what this does, well, it's something that renormalizes the lambda phi 4 operator. So you need a counter term for the lambda phi 4. So you add to your theory that counter term. And you could get rid of this divergence.

OK. So, so far-- hopefully standard stuff. Let's keep going, think about other diagrams. So what if I put in a tau term and a lambda term? This integral is the same integral. I just have different fields on the outside.

So it's got the same divergence, but now the operator it's renormalizing is an operator with 6 points on the outside. So it's renormalizing the tau phi 6 term.

So I insert one tau and one lambda, and I have to get back the renormalization of tau. Well, that's not so bad. We had tau from the start, if we include the tau term, so not really a problem from the point of view of a standard renormalization program. But we could also include two taus, like this, again, same integral, so same divergence.

And now, this renormalizes something that we haven't included yet, something with 8 points a phi 8 operator. So in order to renormalize that diagram and make the theory renormalizable and an effective field theory sense, we need to include the phi 8 operator.

So if I'd ignored the dots that I wrote down-- so let me say, without the dots-- then phi 8 wasn't there. And so then, therefore, I would say the theory is not renormalizable. That's what makes the tau operator the phi 6 operator via non-renormalizable theory in the traditional sense if we include that operator.

That's the classic way of thinking. And the effective theory way of thinking is just that we have to add that operator as soon as this diagram would become relevant. So we just determined a minute ago that tau goes like lambda nu to the minus 2. So tau goes like 1 over lambda nu squared.

And in this diagram, we have two powers of tau. So it's even less important. So lambda nu to the fourth-- downstairs.

And so when it becomes relevant to us, we want that kind of accuracy that we want to include things that go like 1 over lambda nu to the fourth. We have to consider this diagram. And we have to consider adding that operator to the effective theory.

And that's the sense in which we say that the theory can be renormalized order by order in its power kinetic parameter, which is 1 over lambda nu. So I could order this order, but we have to add this operator.

OK? Make sense? Silence means that it makes sense. Jumping up and down saying it doesn't make sense means that it doesn't make sense or puzzled looks from everybody, but that's harder to discern.

So we can summarize this way of thinking in the following way. Remember with the effective theory that we're only interested in computing things to some accuracy. And the accuracy controls where we stop in the series.

So if we're interested in stopping at lambda nu to the r or 1 over s to the r-- s was big. And lambda nu is also much bigger than the p. But let's stick to talking about lambda nu.

And we include all operators that have dimensions up to a certain level. And since the power counting is connected to dimensions, we're kind of guaranteed that we will have everything we need.

So as I promised, what this little argument or discussion tells us, is how power counting is connected to dimensions. And this is the classic way of thinking about effective field theory is that the power counting is connected to dimensions. So this seems pretty generic actually.

You could imagine that, if I did scalars, I mean, fermions and scalars of gauge theory, then I could still go through the same type of arguments, write down higher dimensional operators, go through all these arguments. And so it seems that I'd actually shown you something more powerful than what I claimed. Because I said, here, it seems like it's almost this, right, that power counting is always connected to dimensions.

So can anyone spot where there was an assumption in what we did that where in some case the power counting might not have been related to dimensions? It's a tough question.

AUDIENCE: If the power counting isn't a ratio scale?

IAIN STEWART: Yeah. But I want a little bit more than that, right track. So going back to this example that we did, what did we assume at the beginning that led us here?

AUDIENCE: That we changed the mass point?

IAIN STEWART: No, the mass wasn't so much the issue. So what it was is that we scaled all the coordinates by the same amount. We said all the coordinates are getting large in the same way.

And we could have done something more complicated than that. We could have said some components of this coordinate are getting larger, faster than other components. That's what you do in a non-relativistic theory where the time component and the spatial coordinates would scale in different ways.

So that was the assumption. We assumed, basically, that everything was getting large. All the coordinates were getting large uniformly with s. And we said x mu i was equal to s x prime mu i.

With a universal s for all the components, that was an assumption that led us here. And we may not always do that. In fact, in some of our examples, we won't do that. But here, for the standard model, that's what you want to do.

So if we have the standard model, which is just L0, the usual standard model, then we know, as part of the way of constructing it, that we wrote down all the operators with dimension less than or equal to 4. And we also know that it was renormalizable in a traditional sense.

So now, let's talk about the standard model corrections, i.e. terms in the standard model from an effective field theory point of view that are operators we can write down, like L1. So L1, I can write it in the following way, which is kind of a convenient thing to do. Pull out the scale lambda nu.

Leave over some dimensionless constant. So I'll just use some scale lambda nu for all the operators that I consider. And I'll just allow for differences between the various scales that the operators could have to be taken up by dimensionless constants. And O5 here is a dimension 5 operator.

The dimension of c is 0. In a power counting notation, we say c is of order 1. That means that we don't count any powers of lambda nu associated to c. What we've done here is we've made the power explicit by just writing it in.

So that's often convenient just in the same way it was convenient to make the s's explicit in that argument. Now, we're just building up the theory, writing down operators, making the lambda nu's, which are our power counting parameter, very explicit.

Now, the statement that, in the standard model, it was renormalizable in the traditional sense, I told you that what that meant is that nothing in lambda 0, nothing in Lagrangian 0, really tells us about lambda nu. So we're free to take it as big as we want.

There's no constraint on it from our leading order Lagrangian. In particular, we can take it much bigger than things like the taught mass or the w mass. And we can make these corrections as small as we want.

So L1 is, therefore, really we can think of it as really giving some small corrections. And we can adjust how small they are by just dialing up the scale of lambda nu.

All right, so let's get down to business and actually talk about what this Lagrangian is. Our notation that we index the Lagrangian in a series, this is our sum over n that we talked about last time. And our power counting is that we associated this guy here with no powers of lambda nu, this guy here with one inverse power, this guy here with two inverse power, et cetera.

And we want to think about using this for some p, which you could say is of order m top squared, some scale that's much less than the lambda nu. It could be larger than on top. It could be 10 TeV, whatever we decide. But it's just, in order for me to write something on the board, let me write m top.

OK. So how do we construct L1 and L2? What do we assume? Well, one thing we assume, or we are free to assume and is a reasonable assumption, is that there are not going to be any Lorentz invariant or gauge invariant violating terms.

OK. So we can maintain these as symmetries of our theory. So we assume that they're unbroken. So that means, when we write down these L1 and L2, or generically each Li, that we're going to have to do it by writing down operators that are gauge invariant. Even though they're higher dimension, we still have to satisfy gauge variance of the standard model and Lorentz invariance.

So that's going to restrict what we can do. We also construct Li from the same degrees of freedom that we have in L0. So we know what fields to use.

So that's important. It means that, once you've got the leading order effective field theory, you know where to go for the higher order terms because you're just using the same fields. I'll also make the assumption that the Higgs vacuum expectation value is going to stay to be the value in L0.

So the way that the gauge group is broken by the Higgs vacuum expectation value is spontaneously broken. We're not going to mess with that. And we built into this idea that we do this because there's no new particles that are produced at p.

If there was a new particle produced at p, then they would have to have a mass that would allow us to produce it. And that would mean that it doesn't have a mass up at this lambda nu scale. And we'd have to include it in our effective Lagrangian.

So by taking this point of view, we're assuming that there's no new particles that are produced at the scale p, only at lambda nu. And effectively, we've integrated out, if you want to use that language-- although we're doing this from the bottom up. If you wanted to have a top-down language, you'd say we integrated out the particles at the scale lambda nu.

OK, so that's our logic. So let me just start seeing what we can write down. And for the dimension 5, it's actually very restrictive. Gauge symmetry is very restrictive for dimension 5. And there's basically only one operator.

So we won't stop at dimension 5. We'll go up to the dimension 6. At dimension 5, it turns out that, once you satisfy the gauge symmetry, the unique term that you can write down looks like this.

And my notation is that this guy is our left-handed lepton with a charge conjugation operator. And these guys are doublets. So the Higgs doublet is a doublet like that. And the left-handed leptons, neutrino, an electron is a doublet like that. So these are doublets.

And I'm figuring out how to contract with doublet indices, but I have to satisfy the U1 hyper charged gauge invariance. This has no color. So that's automatically satisfied. And then I have to satisfy the SU2. And I've done that by the way that the indices are contracted.

One thing I didn't write down in this operator is flavor indices. So if you were to add flavor indices, you could do a bit more. But in some sense, that's a pretty simple generalization.

So we'll still count it as one even though we could think about having more flavors. We're contracting things with more flavors. And that would, of course, affect arguments made on gauge symmetry alone.

So in all my counting today, I'll be agnostic about flavor matrices. So when I say only, that proviso is hidden there, OK? So this guy is kind of interesting from an phenomenological point of view.

Because if you replace the Higgs field by its vacuum expectation value, which means getting rid of H+ ? And taking h0 to be a constant, which is v, then that gives a Majorana mass term.

So the observed left-handed neutrino would get a term in its Lagrangian that, after we do that, it looks like that. And that's a Majorana mass term, where this m nu is a parameter that shows up, but it's built out of the parameters that were in this thing. So once I replace these by [INAUDIBLE] and go through the various factors, we get something like that.

So just the fact that we know that the observed neutrino masses are less than, say, 3.5 eV tells us something about the scale. So this is small. If c5 is of order 1, I told you we know what v is. It's 246 GeV. That tells us something about lambda nu that it's big.

But we could try to think up some reasons why the c5 maybe have some suppression in it and make the c a little bit smaller. But if C5 is order 1, then we get a very large lambda nu scale. I'll give you a problem on your problem set to explore this in a little more detail.

I should also note that the Majorana mass term, having two neutrinos like this, violates lepton number, which is a global symmetry of the standard model, at least classically. So this guy violates lepton numbers.

You can also write down dimension 6 operators that violate baryon number. And I'm going to leave that, also, as a problem set problem. So you'll figure out what those operators are.

So example 3, if I conserve lepton number and baryon number-- which are things that, if they're broken, that's obviously having a big impact. And there's obviously strong constraints on that. So you can ask then, if you go to dimension 6 and we conserve those things because they're highly constrained, how many operators are there left over? And there's 80.

So L2, we can account exactly how many operators there are. And there's i from 1 to 80, some coefficients and some operators that are dimension 6. And if I'm going to make a lambda nu explicit, I put a lambda nu squared in there.

So 80 sounds like a big number, but big is always relative. So if you think about 80 relative to, for example, how many soft SUSY breaking parameters you have in the MSSM, is greater than 100, then 80 doesn't sound so bad. Also, you should remember that, if you're going to do some phenomenology with this L2, that many of the 80 are not going to contribute.

If you look at a particular process, only some small subset of them will contribute. And there's many, many, many observables in the standard model with all the different particles. So 80, once you start dividing it up into camps that contribute to different observables, is not such a large number. Or at least it's not an unmanageable number, and people do phenomenology with this.

So for any observable, only a manageable number contribute. And I should also say that, if you have a top-down perspective where you have a new physics theory that you've constructed that has the scale lambda nu in it, then from this point of view what that theory predicts is a particular pattern for the c's.

Hopefully, if it has less parameters than 80, then you get some patterns of connections between the c's. So if you have some new physics model that has the number of parameters that are less than 80, then you get connections between these c's. And you could think that, if you use this Lagrangian, constrain the c's, that you could test generically for classes of new physics theories that are ruled in or ruled out.

Because they have to, if you match on to these c's from those theories, obey whatever constraints you would derive from this logic. Of course, that assumes that the new physics particles are at a high scale. So we can make this expansion, OK? Questions about that? Yeah.

AUDIENCE: [INAUDIBLE] assumption that you don't have [INAUDIBLE] freedom is returning to the high order.

IAIN STEWART: Yeah.

AUDIENCE: If you want them to do a [INAUDIBLE] or [INAUDIBLE] for high order, would [INAUDIBLE] they have to be at higher [INAUDIBLE]. And why is it at a higher energy than [INAUDIBLE]?

IAIN STEWART: So it's not to say that, at higher energy, that no new degrees of freedom would show up if I really probe those energies directly. But what I'm doing is I'm saying that I'm probing the physics at small energy. And the way that high energy degrees of freedom would show up is by a contribution to one of these operators.

So say I have added in some new particle at 10 TeV, mass of 10 TeV. If I expand in momentum over that mass, then what will happen is you'll get an operator like this O6, where that particle is removed because I expanded it. It got removed.

And its mass will be exactly in this denominator. It'll show up here as lambda nu. So think about it as lambda nu squared could just be the propagator. If I had 1 over p squared minus some massive-- I don't know, some gluino, right?

And then I start expanding this. The first term where I drop the momentum is just that. And that could be exactly this lambda nu squared.

So the new physics particles don't show up in Oi6. What they affect is the pre-factor. So you don't need to add new physics particles in order could construct the operators. You're just building those operators out of the standard model degrees of freedom.

AUDIENCE: So you're still working in p but just [INAUDIBLE].

IAIN STEWART: Yes, yes, then the lambda nu. Is there another question? OK. So I've been assuming some familiarity with the standard model here. And I've posted also, as I said last time, my lecture notes on quantum field theory 3. And there's some review reading there if some of this is unfamiliar to you, discussing [INAUDIBLE] and things like that.

OK. So what kind of operators can we have at dimension 6? I'm not going to list all 80, obviously, but I'll list a few of them. So we could take an operator that's the following, built out of gluon field strengths.

So making it Lorentz invariant and contracting up the indices and contracting up color indices with an FABC, that's an operator that's dimension 6 because the Gs are dimension 2. That's one of the 80. You could also do something with fermions.

AUDIENCE: Sorry, what are the [INAUDIBLE]?

IAIN STEWART: Here? Whoops. Yeah, what happened? All right, there we go. Thanks.

So here's something with a lepton doublet and a quark doublet. We already introduced the notation for LL. And QL is similar, but just up and down.

So that's each fermion dimension 3/2. So four of them is dimension six. There's something called magnetic operators.

So there's lots of different things that you can do with four fermions. I've only given you one example. And I've posted the reference for the paper that lists all 80. So you can look at it if you want to look at the complete list.

So we can do something where we have leptons, a Higgs field, as well as a field strength for the SU2. So this is in the SU2. And these are all gauge invariant, as you can convince yourself by looking at the standard model gauge transformations of these operators.

And so if I write something like this, where this is a doublet and this is a doublet and I don't write the doublet contraction, then I'm just contracting those indices. These two here contribute to the mu on the magnetic moment, anomalous moment. So they contribute to g minus 2.

So g minus 2 at the muon, which is something that we've measured to very high precision has what sometimes you would call standard model contributions, which people usually mean as the contributions in our notation from L0 and then plus some contributions from these higher dimension operators, whatever coefficients these operators have.

And again, I would replace the Higgs field here by a [INAUDIBLE]. I know they're dimension 6, so there's lambda nu squared downstairs. One factor of dimension is made up by the [INAUDIBLE]. And then the next scale that comes in is the mass of the muon. So this is the generic size of those contributions.

And again, if you take into account experimentally how well we've measured this, it puts a pretty strong, or at least it puts a constraint, on lambda nu. Actually, it's probably stronger than this number, some number greater than 100 TeV. Maybe it's even 1,000 TeV.

AUDIENCE: [INAUDIBLE] contribution, it was [INAUDIBLE]?

IAIN STEWART: Yeah. So the standard model contribution makes up all the digits we've measured. And then there's some digits where there's some uncertainty. We haven't [INAUDIBLE]. And you can constrain, based on experimental uncertainty, how big this possible contribution could be.

And there's 2 and 1/2 deviations from the standard model in this observable. So you can make up for them with an operator like that. But I never really pay attention to things that are less than 4 sigma personally. Though sometimes it's interesting to get excited about 3 sigma.

AUDIENCE: [INAUDIBLE]

IAIN STEWART: No, I'm not.

AUDIENCE: Oh.

IAIN STEWART: Yeah. So there's also flavor on top of that. That's right. So flavor is actually highly constrained. And you could put in some assumptions about flavor and then you get to the [INAUDIBLE].

OK. So for the remaining 76, see the reference I've posted. So this paper actually wasn't the first to try to enumerate these operators. As you can imagine, this would be a pretty standard thing to do.

But it was the first to get 80. And the reason that they got 80 where other people got more is because they used the equations of motion to simplify the operators. So let me phrase that as, is there a caveat?

So when they did their counting of the operators, they took the equations of motion from L0, and they used that to simplify L1. So they worked out the standard model equations of motion that tree level.

And then they applied those equations of motion to reduce the form of the operators down to 80. So that got rid of a lot of operators for them that other people had considered as part of the counting in the past.

So for example-- so you get some idea, if I have a covariant derivative acting on a right-handed electron. And the equation of motion in the standard model relates that to [INAUDIBLE] couplings. Here, I'm writing the flavor indices in the left-handed doublet Higgs field and the left-handed doublet. This would be like a mass term if I put in the [INAUDIBLE].

So that's like the analog, if I write it in terms of spinners, of p slash u for the right-hand electron is m u, but just written as an equation of motion. So they're using things like this to get rid of covariant derivatives acting on the right-handed electron and just replace it by an operator that a priori doesn't look like it's equivalent, which is H dagger L.

Now, if you are stuck at tree level-- which is actually the language in which they constructed their paper is just to think that you apply this in a way that is valid when you look at lowest order, the operators at lowest order. Then it's pretty obvious, actually that this is OK. If the lines that are coming out of your operator are always external lines, then when you look at the final rule for those lines, you're putting them on shell.

So for example, let's think that we had an operator that was not part of their list. So H dagger H e right id slash e right, you won't find that as one of the ones that's listed in the 80. You could think about the Feynman rule for that. So there's two Higgs particles, say, and two right-handed electrons.

And if I just take the derivative out of here and then I get a p slash and if I use p slash as m u, then I get just u bar u. And so this is, of course, connecting us to the left-handed doublet, the left-handed guy. So this operator, in that sense of thinking about using the equation of motion, which are connecting left and right fields, putting them together in the standard model, is connecting this operator to the one that we just have H digger L.

And you could just immediately get the right result just by starting with that operator. And that's the logic that they used. So you don't have to write this down because it's redundant. You get the same result from writing this down.

So that seems fairly straightforward at tree level and at lowest order where all the fields are external. But I actually claim that it's true regardless of what I do. Whether I have loops, whether I have propagators, it's OK. We can do this. And that seems a lot less trivial.

So they didn't know that when they wrote their paper. At least I don't think they did, but it is true. So that's what I actually want to talk about for the rest of today's lecture because it's a pretty powerful thing to do.

If we are allowed to use these equations of motion to simplify the form of the higher dimensional operators in our effective field theory, it certainly helps us to reduce the number of operators.

So one way of phrasing this is what's called the representation independence theorem, sometimes called that. I'll phrase it a few different ways, but I'll start with this way. So if we have some field, phi, we can set it equal to some combination of other fields.

So phi and chi can be scalars, let's say. And this function, f, which could be fairly complicated, has to have at least one property that, when chi is 0, it's 1. So phi and chi both show up linearly in some term in the function if you'd like, if you want to think of it as a Taylor series, for example.

So if that's true, the statement of this theorem is that calculations of observable that are done with phi or the Lagrangian that's made of phi-- and what that really means is that I've quantized phi-- will give the same results as those with a Lagrangian where I quantize chi where I construct that Lagrangian by just making this change of variables.

And we're going to exploit that fact in order to argue that we're allowed to do what I just said. So we'll start slowly, then we'll build up. Then we'll state a more general theorem than this one. Then we'll show you how to prove it.

So let's start by thinking of an example. Examples are always good. And we'll stick with our scalar field theory. I'm getting tired of writing factorial, so I'm choosing a little bit of different normalization here.

So I'll consider an effective theory that has these terms at least to start. And eta has dimensions. And it's a small thing. It's like 1 over lambda nu.

And the statement that we want to explore is the fact that we can use the equation of motion to effectively drop the last term by using the equation of motion. So how do we make use of this statement?

Well, this tells us that we can make changes to variable and that we won't change anything. So we try to make a change of variable to make this go away.

AUDIENCE: Wait, sorry. But that's not even the tree level operation [INAUDIBLE] right? You still have the lambda term.

IAIN STEWART: Yeah, that's true. Yeah. Yeah. I should write the lambda term in there, plus, minus, whatever it is. Thanks. There's a 2. Thanks.

All right, so how do we to get rid of this last term? Well, let's just use our theorem by making a field redefinition. So I claim, after making this field redefinition here, that something magical will happen, or something that we want will happen.

So we're going to integrate by parts at will, which is often a convenient thing to do when you're making these field redefinitions. So the term that's a 1/2 del mu phi squared goes to /2 del mu phi squared from the first term. And then we pick up a term that's exactly in the form we want to kill this extra term here that was proportional to del squared.

And then there would be order eta squared term. We have to consistently make that field redefinition everywhere, so do it at the mass term as well. Well, let me write out the eta term here, just a eta squared term just so you have all the terms.

And we keep doing that. So we would do it also for the lambda phi 4 term. And we do it for this phi 6 term.

And what you get out when we do that is that you can write down the Lagrangian. And you can group together the terms. And that operator is gone.

So it got cancelled by this guy. There's also these other things that are induced, but really what that does is it gives you more terms that are phi 4 and more terms that are phi 6. So you can really think of the other terms as just this guy here is just adjusting the constant of the phi 4. So if it's no longer lambda, it becomes a term that has this extra piece to it.

So I'm going to give you more on this on the problem set. On the problem set, you'll work out explicitly with these relations are. And I'll also, on the problem set, let you think about what's going on here when you start considering loops.

In this particular example, I'll ask you what's going on. I'll ask you to show what's going on and that it's still OK when we have loops. So rather than explore this example in more detail, which I'm asking you to do on your problem set, let's state a more precise definition of what we're doing here. So that's the idea, that I can make a field redefinition.

When I make a field redefinition, it's always going to do something from the kinetic term because the kinetic term is there. If there's a higher order term that's proportional to that kinetic piece, then I can set up my field redefinition in order to cancel it off.

So there's some things that are important here. We have to make a field redefinition that preserves symmetry. So if we make a field redefinition that breaks Lorentz invariance, you can't expect that you're going to have a Lorentz invariant description after that.

And this statement of f of 0 being 1 is the statement that you have to preserve the same one-particle states before and after the field redefinition. And such field redefinition definitions basically allow the classical equations of motion to be used to simplify the theory.

I'll also put the proviso in that it should be a local quantum field theory. And there is no statement in this theorem that we have to stop without considering loops or without considering propagators that can really make this argument hold even beyond the level that we've showed it, but with loops and with propagators in as well.

So there's some references. Again, I've posted the one that is closest to our discussion, which is this paper by Arzt. There's also a classic paper by Howard Georgi.

The title of the paper is "On-Shell Effective Field Theory." But I'll mostly follow the notations in this paper by Arzt in our discussion here. We'll go a little further than he does and elaborate a little more, but it's basically the same notation.

OK. So how do we prove something like that? Well, there are some lessons in this proof. So it is something that's worth going through.

So the way that I set up my example up there, I was power counting in eta. Eta was 1 over lambda nu. So let me just write our effective theory organized as in a series in eta.

And let's consider removing some operator that looks like we should be able to remove it by making a field redefinition. And I'll try to be a little bit generic about what the form of that operator is. So I'll say it's covariant derivative squared acting on a scalar field just, again, to make things simpler.

But then multiply by any function of all the other fields in the problem, and that's what t is. So phi is a complex scalar. And t is any function that sort of meets the needs of our symmetries of the problem.

And I'm just using this other phi as a shorthand for all the other fields that we might consider. So that could be fermions, other scalars, gauge fields. So let's say we want to get rid of an operator like that.

Oh, I should also say that it's local. All right, well, let's write down the generating function for this theory, path integral over the fields exponential. I'll regulate the problem with dimensional regularization.

It's convenient. It preserves the cemeteries. That's something we want to do. I write out some terms in the Lagrangian. We won't need L2. We'll stop at L1.

But the idea, if we were to think about L2, would be similar. And let me write it as adding and subtracting something. So I'll subtract td squared phi and then add it back.

So this is, if you like, you can think that L1 had a td squared phi. And what I'm doing here is removing it. So this is what we want.

OK. So I'm just making it explicit, but still writing things in terms of L1. So there's that. And then there's the coupling to the source. And kind of in a generic notation for each field phi k, I have a source jk.

And I truncate everything at order eta. OK, so that's my starting point. And then Green's functions are obtained by functional derivatives with respect to the j's.

All right, so what we're going to do, we're going make a change of variable in the path integral. So of course, there's additional complications beyond what we were doing when we were just thinking about it at Lagrangian level. There's basically two additional complications.

When we make a change a variable in the path integral, we're going to change the Lagrangian. We're also going to change the measure. So there could be a Jacobian. And we have to worry about what happens with the source term.

So let me just think of it as a change of variable on the phi dagger. That's why I wanted to think about a complex scalar, so I could think about it as phi and phi dagger and just make the change of variable in the phi dagger. But doing a real scalar is just-- everything would go through just as well.

So there's one term, which is the phi dagger term where there's a Jacobian factor. So here's that promised Jacobian for making that change of variable. Since the change of variable is order eta, it's not affecting L0.

And I can write, in a kind of nice notation, the way that these eta t terms from this change of variable show up. I can write them as taking all the terms in L0, looking for a phi dagger. That's the derivative. And then I replace it by an eta t.

Now, some places in L0, there won't be just a straight phi dagger. There might be a del mu phi dagger, right? So I can take that into account as well by taking the functional derivative with respect to del mu phi dagger.

So any del mu phi dagger term, if I integrate by parts, and then take the derivative, you can see why this is the right form. Integration by parts is what gives the minus sign. OK, so this finds all the phi daggers in my L0 and sticks in an eta t to order eta, which is the order we're working.

So anytime we had in eta, which is these terms here, then we would induce something ordering eta square. And we're dropping those terms, so that's not something we have to worry about. And then there's the source.

And one of the sources is 4 phi dagger. So for that particular source, j phi dagger, we induce a term that's j phi eta t and then plus order eta squared.

So as I said, there's three types of changes. There's a change to the Lagrangian, like in the example we did before. There's a change to the Jacobian through the Jacobian.

And then there's a change through the source. We have to worry about whether any of these things will matter. And basically, what the claim is and where we're going is considering the Lagrangian is actually enough, so that 2 and 3 are something that we can kind of deal with generically. And then we only have to check that changing Lagrangian does what we want.

So the claim is and what we'll show, without changing the s matrix, we can remove considering 2 and 3 rather generically. So we don't have to worry about them.

So some of this we'll do next time, but let's just start today. So let's first look at del L, kind of analogous to what we were doing in our example before. So what we need is a change of variable, which is this one up here.

And one restriction on del L, which I mentioned as part of the assumption, was that this change of variable should transform in the same way as the original phi dagger, so same Lorentz index structure, same gauge index structure, et cetera. So that's kind of an assumption that we said.

In order to respect the symmetries and not mess them up with our field redefinition, we assume that. So let's see what happens to L0.

So there is L0. Maybe it has some other terms as well, but let's just consider these pieces and ask what happens with them since they're the relevant ones. So when I switch to prime fields, I get that back again.

I get m squared phi dagger prime phi prime. And then I get some new terms, which I integrate the covariant derivative by parts, which I'm always able to do. It's nice what gauge symmetry allows you to do.

Then I get a term like that, and I've set things up so that this term here does exactly what we need. I go back over here. Those terms are cancelling.

OK, so that's good. Now, you may be worried about all the other terms that get induced. You've removed something, but you've induced a lot of stuff.

But the point of the effective theory is that you already wrote down every possible operator that was consistent with the symmetries. So even if you induce a bunch of other terms, they should be terms you already have. So all you're doing with inducing those other terms is shifting the coefficients of the theory around.

So L1 was already complete in the sense of having all the terms allowed by the symmetry. And we've respected the symmetry in the way we've made this field redefinition.

So any terms that I didn't write, which are all these terms in the da, da, da prime are already operators that are present in the other dots that I didn't write. So there were some dots here. There was some dots there. Even if I include operators in those dots, the operators in these dots are already present in those dots. And so all I'm doing really is shifting couplings.

OK. And that's same effective theory just with a new name for the coefficients, but we haven't fixed the coefficients yet anyway. We're setting up our effect effective theory. So whether we fix the new coefficients or the old coefficients, it's perfectly fine.

So number two is the Jacobian. And I won't go all the way through this, but what are we going to do there? We're going to use the same kind of trick that we did in gauge theory.

We're going to write the Jacobian as a Lagrangian involving ghosts. So remember, when you talked about Faddeev-Popov in some field theory course prior to this one, you saw that you could write a determinant as an exponential involving some ghost fields. And the way that it worked is that, up to a sign, whatever was sitting here ended up just sitting between the ghost fields.

So this is the ghost of Faddeev-Popov procedure. And we're going to just use the same thing, although [INAUDIBLE] we're talking about scalar field theory. So for us, we take this guy, which is the Jacobian, I'll take that derivative.

And if you take this thing, which is sitting in functional determinant, and you turned it into ghosts, then what you get is a terminal Lagrangian that looks like that.

So that looks like it could do something. And I think I'll take this up next time. It turns out that this term is actually a ghost that has mass that's of order the high scale, lambda nu.

And so, again, because of our logic of the effective theory only being valid below that high scale, we can just effectively integrate out this ghost. And then it shifts coefficients again, but I'll continue with that next time. So we have to figure out what to do with this ghost Lagrangian. And we'll deal with that next time.

So any questions about our halfway done proof? No? OK. So this is a pretty powerful thing that you can use in effective field theory. And it also keeps your eye on the ball because it tells you to think about physics.

Because this leaves physics invariant. It doesn't leave something like if you-- say you had a theory that had some long distance degrees of freedom. And it had some short distance potential.

The short distance potential could be changed by a field redefinition. It might not be a physical thing. It keeps your eye on the ball as to what is physical and how to talk about things.