PROFESSOR: Apparently. We'll start slow. So last time we were talking, we just started talking about effective field theory with a fine tuning. And what actually that means takes a little bit of discussion. So what you could mean by a fine tuning is that you have something that's irrelevant. You look at the operator, you think it's irrelevant, but it's not. It's relevant. Something that you should include even at lowest order in your power accounting.

But saying something is irrelevant means that you have a power counting, that you understand the power counting for the theory, what the correct power counting is. So in this example that I'll give, what "irrelevant" will mean is that you basically do a dimensional power counting, which is how we usually think of defining irrelevant and relevant. Do a dimensional power counting and you end up finding an operator that-- you find that the operator looks irrelevant, but when you do calculations you can see that should be relevant. And that means, really, what it means is that this natural power counting of dimensions is not the right one, and you have to do something more complicated.

But it is still is a sense in which it can be thought of as a fine tuning, because as you'll see, the changing of the power counting from the naive one to the more complicated power counting involves some kind of tuning, if you like. And it really, in this case, we'll see actually that it corresponds not to expanding around the trivial fixed point, where you would have a free theory, but expanding around an interactive fixed point.

So it'll be a little non-trivial, but we'll be doing this in the context of two nucleon effective field theory. And the advantage of this is that the nucleons are going to be non-relativistic. So P is going to be much less than M pi. It's going to be a very simple theory. Everything that's an exchange particle gets integrated out. It's just a theory of contact interactions, and derivatives of contact interactions. And because it's non-relativistic, we can actually calculate all the loops to all orders and perturbation theory, and we'll do that in a minute.

So this theory, we can calculate a lot of stuff. And so we'll actually be able to see how this non-trivial fine tuning works, and explore it from multiple directions, and we'll be sure that what we're saying is actually correct. So it can be a lesson for understanding some of the concepts and other effective field theories with the fine tuning, which you might want to design, where you don't have as much ability to calculate.

All right, so let's start off with something simple, which is elastic scattering. And that's mostly what we're going to talk about. Two particles in, two particles out. Center of mass frame. They scatter to some P's coming in, P primes going out. And this is basically a problem that you could treat with quantum mechanics. It's like non-relativistic scattering. So if you have a single partial wave, then this scattering is described by a phase shift, delta. And the relation of the phase shift to the amplitude, with our normalization for the amplitude, is this. So this is the S-matrix, it's just a phase, and that's the relation of the S-matrix to the amplitude. And this thing is the amplitude. And I guess the other thing we know is that, by energy conservation, the magnitude of P is equal to the magnitude of P prime.

All right, so if we rearrange this equation, and we write it as A and put the phase, solve for a. Could do that. So that gives that equation, which we can rearrange a little further. Now, there's kind of a conventional way of writing it, which is that the amplitude should be given by 1 over something that's P cotangent delta. Scattering angle, or that S-matrix angle, and then minus this IP. OK?

So the I of-- the I's just-- shows up here in this part, and that's the complex part of the amplitude. That's basically going to be related to unitarity, that that IP is there. This S-matrix is obviously unitary. This digress is 1.

All right, so let me tell you something about non-relativistic scattering, which on the face of it, looks kind of non-trivial. So this thing, P cotangent delta. So here, I was doing a single partial wave. Yeah, OK, no, it's fine. So what is this L? This L is the partial I am considering. S wave, P wave. So L is 0 for the S wave, L is 1 for the P wave.

And the statement is that if you have a short range potential, and you pick a wave, then this P cotangent delta with the appropriate power of P can be a Taylor series expansion of P. OK? And this is actually something that's quite difficult to prove in quantum mechanics, this particular fact. And it's called the effective range expansion. It's difficult to prove because when you do quantum mechanics, you pick a potential. And I'm saying that this is true for any potential. So if you start doing quantum mechanics with some potential, you've got to prove that you can put it in this form irrespective of what the choice of that potential would be. And that makes it a little bit tricky to do in a quantum mechanical setup. But we'll see actually that this is very easy to prove from an effective field theory setup.

So, as a way of getting into this effective field theory of two nucleons, let's prove this fact. So what is the Lagrangian for this theory? There's no-- it's not a gauge theory, so we just have ordinary derivatives. So if you like, you can think of what I'm writing here as kind of like our IV.D term, except I think the center of mass frame, and this would be like the kinetic energy term, but now it's just partial squared with no D, et cetera.

And then there's a bunch of contact interactions. So there's a whole bunch of operators that are involved in nucleon and fields with some Wilson coefficients. The notation here is this S is kind of a pseudonym for the channel. So this S here-- maybe it should be a script S or something, is telling me what channel I'm in, and if in a spectroscopic notation, you'd say you're in the 2S plus 1 LJ channel. So this would be the angular momentum, total momentum in the spin. And these operators here, for our purposes, are four nuclear fields with 2M derivatives.

Now, this is not really the complete theory for a couple of reasons. Well, there's higher order relativistic corrections indicated by the dots. There would also be dots over here That could have to do with having more nucleon fields. For example, I could have-- instead of just four, I could have six. But I don't need to worry about those operators for two-to-two scattering, so I'm leaving them out. So this is actually the complete theory, if we include these dots here, for two-to-two scattering.

And this nucleon field, [? its ?] spin [? at ?] half, and it's isospin been at half too, so it includes both the proton and the neutron. Nucleons are fermions, and that implies, actually, a relation, because the wave function has to be anti-symmetric. And so actually, you know that you can associate isospin and the angular momentum in the following way because of this fact. So all the isotriplets have -1 to the S plus L even, and the isosinglets have -1 to the S plus L odd. So that cuts down by a factor of two the number of combinations you have to consider.

And basically, what this theory has is for some given channel in and some given channel out, which I could denote in general different, we get operators that just will have some power of the center of mass momentum. P to the 2M. And actually, just by angular momentum conservation, that has to be the same as that. And so all that can change is the L's.

And so if S is 0, S is either 0 or 1, because we have 2 spin half particles. If S is zero, L will be equal to L prime because J is equal to J prime, and there's no shift of the spin. So that's one possibility, and if S is 1, then-- so S here is the same, if S is 1, then you can have L minus L prime which is 2, so, or 0. OK? You're shifting by one unit, and you can compensate either by having minus L prime be 0 or 2.

OK, so we conserve J. So we can enumerate all the possible partial waves. We'll mostly focus on the S wave. So, let me write out some of these operators for you. So the first operator has no derivatives, and I can write it in a way that makes the partial wave explicit if I do the following. And then there would be some derivative operator. And I'm going to pick the normalization to make our lives as easy as possible, as we usually do when we're setting up the operator basis.

There's the first two guys where this derivative operator is like, [? grad squared ?] to the left. [? Grad ?] left dot [? grad ?] right. That [? grad ?] go to the right. And these P's, if you look at them, they're just matrices in the spin and isospin space. So the two we're focusing on are the S waves. And in the S wave you either have 1S0 or 3S1.

And so we've encoded all the, sort of, complexity in just these matrices, which kind of just go along for the ride, and I'll tell you what they are. So I sigma 2 projects you onto a spin singlet, and I tau 2 tau I projects you onto an isotriplet. And then likewise, 3S1, which is a spin triplet, you put I sigma 2 sigma I. I tau 2. I tau 2 and I sigma 2 are just because of the way I wrote the operator. I wrote it, instead of writing N dagger N, I wrote N transpose N, all dagger. And that means, basically, you should think about the way this operator works, is it annihilates two nucleons in a particular spin wave, or a particular spin, isospin channel, and then creates them again. So annihilate, create.

So I just put the two fields that are doing the annihilating together, and the two fields that are doing the creating together. And that's nice because you're annihilating them in a particular channel. So with those conventions, our Feynman Rules are particularly simple. If we just have a C0 in some channel, then the Feynman Rule is just minus IC0, and if we have one of these higher C2 operators in the center of mass frame, it's just minus IC2P squared.

In the center of mass frame, that's the center of mass momentum. And remember, in the center of mass frame, P squared was equal to P prime squared. And so we can actually just write down the Feynman Rule for the complete set of operators there if we adopt this convention. So if you insert a guy with 2M derivatives-- derivatives always have to come in pairs because of angular momentum.

You just have that Feynman Rule, sum's over the number of derivatives. So this is the complete, in this theory, this is the complete tree-level amplitude from those interactions over there. It's very nice theory. Simple. What about loops? We are going to need some loops.

So let's look at the following loop, and I'll start by looking at just E equals 0. Let's just take it, take the nuclear arms and then scatter them again. So, in terms of the momentum flow I have some Q going this way, and then I have minus Q going that way, that's my loop momenta. So I get 2 coupling C0. Want these to be 0's.

Ergo, DDQ. And if I just kept the HQET type term in my kinetic term, then it would look like that, OK? So this is just-- for the moment, if we just keep partial DT in the kinetic term, which is what we were doing in HQET, then we would get that. And that's a bad integral. It's got a pinch singularity. It's an ill-defined interval.

So just from that little algebra, we see that actually keeping just the partial D by DT in the kinetic term is not the right thing to do for this theory. And that's because the kinetic energy is a relevant operator in quantum mechanics. Whenever you write down the Schrodinger equation, you kept it. So the right power counting and should have E, which is of order P squared over 2M. So the partial T term and the [? grad ?] scored over M term should be the same size.

So this is generically true of two heavy particles. They have a different power counting for the kinetic term, than HQET. Any two heavy particles, whether it's too heavy quarks, two heavy nucleons, two heavy anything. All right, P should be [? order ?] P squared over 2M.

AUDIENCE: E being the kinetic energy?

PROFESSOR: So really, what I mean here is just the partial T term in the action over there should be the same size as the [? grad ?] squared over 2M. Yes, I have pulled out the mass. Yeah. So just like in HQET, to get this partial T we pulled out the mass, and we have a kind of non-relativistic type expansion. And the difference here is that we need the partial T to be of [? order grad squared ?] over 2M, and that leads to effectively counting velocities rather than-- because you have to count energies different than P's. We won't spend too long talking about that because we have other things to discuss here, but this is a whole interesting subject in itself. The power counting and what it means.

One thing that's kind of interesting here which we won't cover, which I can't help but mention, is that say you did quarks, which was a gauge theory. This is not a gauge theory that we're talking about, but let's-- but you could do a gauge theory that has this type of power counting and it has exactly the same problem. Just replace these heavy nucleons by quarks, and replace this dot here by cooling potential. Exactly the same problem if you try to use HQET. And in that theory, too, you need E to be of order P squared over 2, and that's called non-relativistic QCD.

Or you could do heavy electrons, where you have QED as the gauge theory. Non-relativistic QED, same issue. E has to be of order P squared over M. And in gauge theory, there's even a further complication, which is basically that there's gauge particles that want to talk to E, and there's gauge particles that want to talk to P, and those are different sizes. So you have something called ultra-soft photons and soft photons that are the gauge particles for E, and the gauge particles for P. Kind of an interesting theory. We want to have time to talk about it. Somebody wants to talk about it for their project, it's kind of fun.

OK, so we have to keep P squared over 2M. At least knowing a little quantum mechanics, we know that that's true. Or running into this pinch singularity, we see that trying to do something different than that leads to problems. So let's redo our calculation here, but now keeping that term, and see what we get. Same bubble diagram. Let me send in on each of these legs E over 2, so E is the total energy that I'm sending in. Convenient normalization. Let's see how having the kinetic energy fixes this pinch pole.

OK, so if you look at the poles in the complex plane here, what's happened is you've split them like this. So that you've moved them along the real axis. And so now you can just think of a contour, for example, if you want to think in the complex Q0 plane, you can think of doing a contour integral like that. Everything is well defined, convergence at infinity, everybody's happy. So we can close above, pick the polar. Q0 is E over 2, minus Q squared over 2M, plus I0. Plug it back into the other one.

My notation is that N is going to be D minus 1. So when I do one of the integrals by contour, I go down a dimension. I'll call that N. So it's N here. This integral we can do. You have to be careful about the epsilons. Because they tell us tell us whether it's minus IP or plus IP. ME is set up in my convention. ME is P squared.

So this is giving a P, but it's giving either a minus IP or a plus IP, depending on the sign of the I0, but I know it's a minus IP. So I used dimreg here. Because if you look at this integral, there's three pairs of Q upstairs, two downstairs. So it's power law divergent. But we don't see power law divergences in dimreg, we just get this finite answer. And actually, that finite answer is exactly the imaginary part that you need if you want to cut to graph, and say that the cut of the forward scattering is the same as this amplitude squared. So it's exactly, in essence, this is the piece that you need to be there by unitary.

If you were trying to keep the E, you could think, well, maybe if I just kept the E in this calculation, it would solve this pinch problem. But it doesn't really do it because if you really stick with the partial D by DT as your leading order action, then the equations of motion are equal 0, so. You have to take equal 0 to go on [INAUDIBLE]. So you can't really avoid the pinch that way, you really have to take this kinetic term. you have to take the kinetic term to have both the partial DT and the [? grad ?] [? squared ?] over 2M. OK, any questions so far?

All right, well there's something here that might bother you. We've got an M upstairs. M is big. M is the mass the nucleon, and it's appearing upstairs. That's always a bad sign. Usually a bad sign. Well, at least that's something we should worry about. So let's figure out where all the M's are. Let's count all the M's in the theory, holding the momentum fixed. So if we hold the momentum fixed, then [? grad ?] has no M's. But partial T scales like 1 over M. And correspondingly, T scales like M. OK. X and [? grad ?] don't scale but partial T and T do. And that's exactly what makes these two terms the same size.

So we can ask, what is the scaling of our nucleon field? And if we want to do that, we go back to our action and we just say the action shouldn't have any scaling. T has a scaling, so there's an M in here. This guy here, the whole thing scales nicely, 1 over M. From what we just said over there. This M cancels that 1 over M, so this guy is therefore M to the 0. The nucleon field has no scaling with them.

So then, once you've done the kinetic term to figure out the scaling of the nucleon field, you can do, then, any other interaction. So that the power counting, the kinetic term, is always which determines the power counting of the field. That's true in any theory, any effective theory, because you're-- that's the basis of the fluctuations you're describing by that field. And once you've fixed that counting, you've got all the counting you need, and now you can go count other operators like the one with 2M derivatives. And this guy here just has-- you can have just vector derivatives, no time derivatives, just vector derivatives, 2M of them, and nucleon fields. So there's no M's in there, that's M to the 0.

And this guy here is an M. So this guy here is C2M, must be a 1 over M. So therefore, any C2M scales like 1 over M. And now you see it's not such a problem, because here I have 2 C0's, each one of them scales like 1 over M, I pick up one more M in the numerator, and that just makes the whole thing feel like one over M. Which is the same order that the tree level was scaling, the tree level with scaling like 1 over M. So loop diagrams and the tree level both scale like 1 over M, and that's actually generically true because every time I add a loop I add a coupling, and so those two M's can compensate each other. So there's no issue with M's.

So the one that graph is the same size as tree level, and they both go like 1 over M. If you now do dimension counting, you can say given that we've identified that the C has an M in it, if you ask about the dimensions of C, -2 minus 2M. Because the dimensions of the nucleon field are 3 halves. And we're doing some expansion in P much less than lambda, so just by dimensional power counting we expect that the coefficients would be of the following size.

We know that there's an M and that it's the only M, and all the rest of the dimensions you should think of as being made up by the stuff you integrated it out. So it could be the pion, for example, setting the scale lambda. So what you'd expect for the C2M's is that this is how big they are, and that the derivatives are then being suppressed because of these lambdas and you're expanding for P much less than lambda. OK. Are we happy so far?

All right, and we'll see when you do matching calculations, this M [? law's ?] always there. And this is just a nice, elegant way of figuring that out. Because this is a very simple argument. All right.

AUDIENCE: I don't know if I totally understand. How do you know that C2M--

PROFESSOR: Goes like 1 over M?

AUDIENCE: Yeah--

PROFESSOR: So the kinetic term-- so at first you know this, right?

AUDIENCE: Yeah.

PROFESSOR: Yeah, so I see. You're worried about whether there could be 1 over M squared corrections?

AUDIENCE: Yeah.

PROFESSOR: Yeah. In principle, there could be 1 over M squared corrections from relativistic corrections, so that this would be the leading order term. Yeah. Not worse than that. If you looked at the P cubed, P to the fourth over 8M cubed term, that term would have a higher power of M, and you could imagine that there's some relativistic corrections in the four nucleons as well. Yeah. So we'll work basically here, at lowest order in the relativistic corrections so that you could put in the relativistic corrections, as well.

All right. So we're basically stopping at that order, which is equivalent to quantum mechanics, but we could go further if we wanted to in this effective theory. So now, let's think about other loop diagrams in the theory, without relativistic corrections, just with these interactions. And let's insert the 2N and the 2M derivative operator on these vertices. You see all the loops look like this, they're all bubbles. And that's the same reason in HQET that you don't have any diagrams that kind of-- you don't have diagrams that look like this, because these types of diagrams involve antiparticles and we only have particles.

So that's the beauty of a non-relativistic theory, you don't have any diagrams like that. You just have diagrams like this. And so basically, the whole theory is bubbles. The theory of bubbles. If you go through this loop diagram, you do the pole the same way. It's exactly the same two propagators. Then you get the same Q squared minus ME, minus I0. And the only difference is you get powers of Q in the numerator. And this, so this integral is one of the ones that shows up in here. And this integral, you can do the same kind of trick that you used when we were discussing field definitions, where you basically take the top and organize it by adding and subtracting.

So add and subtract any like that. And now, you can think, N and M are integers, just expand this thing out. Some number of these factors, some number of these factors. But any time you get one or more of these factors, it cancels the denominator and then you end up with scaleless integrals. So basically, that means you can throw this piece away.

So higher order derivatives actually just lead to ME's, whether they act inside or outside on the nucleon fields, they lead to P squared. Yeah, sorry. This M is a little M, and this is supposed to be a big M. Oh, N. Oh yeah, I am also using that too. Sorry. Yes. This is an integer. I should call it J or something. And the other M is 3 minus 2 epsilon.

Yeah, it's dangerous. All right. But this means that basically, the graphs in this theory are very simple. So if we actually now consider the complete set of diagrams, or we impose-- we put the full amplitude in there and we consider these bubbles. Because of this fact that I just told you, which doesn't change when you insert more bubbles, the bubbles all decouple from each other. The contact interaction is decoupling them from each other. So here's a complete amplitude with K insertions of the coupling, and K minus 1 loops.

OK, so we just completed all the loop diagrams the theory, and it's giving us something that we can sum, its a geometric series. So this is for K minus 1 loops, we can sum them up. So if you like I should have called this the K-th term. And if I sum up, cancel the I on each side, you get that. Let me round it again. Like this. So dividing out the numerator, rearranging the four pi over M's little bit, I can write it like that. And then we can identify this thing here as P cotangent delta.

OK, so P cotangent delta, from our effective theory, we calculate it to be the sum over this, P2M, where now I've conveniently defined it, hatted coefficients, which are the following things. And that just cancels the M that's in the C2M's, so C2 hat scales like M to the 0. So that's just a-- is an obvious way of reorganizing things given the factors of 4 pi over M that we knew had to sit out front of this S matrix calculation of the amplitude.

So we can identify P cotangent delta as something that doesn't involve any M's, and that actually is also what we would expect for non-relativistic scattering. So then you can look at different waves, and you can look at this formula, and you can-- doing two of them will show you how it works. So S wave is L equals 0.

So that's something we can do a Taylor expansion in P in. And this is what the Taylor series looks like. And this has exactly the form that we wanted. This is 1-- some constant 1 over A, which is the scattering length. Some constant times P squared, which is called the effective range. And we see that expansion coming out, and we never had to specify what the potential was, because the effective theory was agnostic to what the potential was. That's the whole power of effective theory, that you don't need to know what the particles were that you were integrating out, they just give you some values for these coefficients. And those values exactly become the effective range and scattering length in this theory.

AUDIENCE: So this seems very [INAUDIBLE] dependent.

PROFESSOR: Yeah, we're going to talk about that. Yes. Yeah, that will encompass the second half of lecture. So we'll come to that momentarily. Let me just do one more way, just so you see. L equals 1. in the L equals 1 case, there's no C0 hat, you need the derivatives to correspond to the wave. And so in this case, you look at PQ cotangent delta, the denominator starts at C2. And the reason why there is this P to the 2L plus 1 is just so that you get the extra P here, which compensates the P's downstairs, and again this thing has a Taylor series.

Same story as over there, we can identify the scattering link for the P wave as the 1 over C2 hat. So all the higher partial ways work the same way. And you just have P to the 2 L plus 1 in front of your cotangent delta. OK. Here you can see how this generalizes [INAUDIBLE]. So that proves this non-relativistic quantum mechanics theorem.

And we did it without having to specify what the potential was, because the potential is encoded in BC's and effectively that's like a basis expansion of the potential. But it's a fun one because it's in delta functions and derivatives of delta functions. So we're doing a local effective field theory, which is not something you'd think of ever using for a basis-- well, maybe you would, but most people wouldn't think of using basis of derivatives of delta functions for quantum mechanics. OK.

Well, you'd hope we can do a little more than quantum mechanics, right? So you can think, if you like, in terms of determining the values of the C's, you can say, well, experiment tells me the values of the scattering length are 0, and that's indeed true. So this equation that C0 hat is equal to A, you could view this as a matching equation. Putting back the M, you have that.

And then for C2 hat, we have r0 over 2, a squared. So pretend that experiment gives r0. And a, which they do measure, and then you know the value of your Wilson coefficients and you can start using this theory.

Now, if you think about the power counting, if a and r0 are order 1 over lambda, that's the natural thing you'd expect. Then you reproduce what we expect. Whatever it was a minute ago. 2M plus one. So if all the constants are scaling like whatever their dimensions are, in this case they're both dimension minus 1 in momentum units, then you would reproduce exactly with this power counting that we said. OK, so everything would be nice.

But nature is not so nice. Or nature threw us in a different direction. When we actually look at the value of these constants, the scattering length is large. So C0 is large, from this point of view. Let me quote some numbers for you.

So in the 1S0 channel, which is the larger one, this guy is 23 fermis. And in the 3S1 channel, it's 5 fermis, and both of these are actually large. We know they're large, look at the errors. If you want to think about momentum units, you could take 1 over a. And for this guy here, taking 1 over a is giving him, like, if I did the calculation right, minus 8.30 MeV. Oh, sorry, no. That's this guy.

And this guy here is giving 1 over a, is 36 MeV. So if you thought the natural size was the pion, then you'd say, well, these constants should be 1 over pi-- that these numbers that are in MeV should be in pi. And 8 MeV is a much smaller number than 138 MeV. 36 is also a smaller number than 130 MeV. So both of these guys are not natural. In particular, this one you see it's very not natural. So there's a fine tuning going on. Some kind of fine tuning from the perspective-- from our dimensional counting EFT point of view. There's a fine tuning that's making the a big.

And if you look at the other guys, the r0 and the other guys, they are exactly of the size you'd expect. So there's no fine tuning there. The only fine tuning is in a. And not the other ones. Just by comparing two data. OK, so how do we deal with that? It looks like we set up a defective theory. It has seems like a beautiful theory, it could describe some things about quantum mechanics, but then we learned that our power counting sucks.

So what we want is actually a power counting it's a little different. Where AP is of order 1, or even AP much greater than 1, we'd like that to be allowed. Where our 0P is much less than 1. We'd like to be able to take a scattering length effectively into account to all orders, and that means basically that we want to treat C0 as relevant. We don't want 8MeV to be the limit of the lowered-- the thing that goes downstairs when we're making a power expansion, right?

We'd like something like M pi to be downstairs. If we want M pi to be downstairs, you got to treat AP to all orders. And then you're just limited by M pi which is the r0 term. And that means you've got to promote C0 from being irrelevant, scaling like 1 over M lambda, to something that's relevant.

So this is actually a problem that occurred because we just proceeded and started calculating, and we used effectively the MS bar scheme in dimensional regularization. We saw powers and fractions, we did nothing. So let's try another scheme. It's a little more physical. Called offshell momentum subtraction.

So what is offshell momentum subtraction? It says take the amplitude, and in this non-relativistic theory, you take P to be at some imaginary point so that you avoid any cuts. And you can define-- and whatever channel we're in, in this new r scheme, you can define the amplitude of that particular point to be the tree level result.

OK, so any loops, if you take them at this point, you should get back that result. So this is the analog of what you do in a relativistic theory where you would take P squared to be minus mu squared. In the non-relativistic theory, it's always P that's showing up, which is the three vector P, magnitude of the three vector. So we should assign some rule for that. And it's just, we take it to be I times mu. So how does this change our calculation? So let's go back to this one calculation.

So now there's going to be some counterterm needed. It's going to be a finite counterterm. Let's see what it does. So the loop graph gave this IP, that's what this graph gave. And this has to be just such that if I set P equal to I mu, that I get 0. So this has two plus mu.

So the counterterm is giving mu r, C0 squared. And that exactly makes this amplitude vanish at that point, and that's what you want because the tree level graph of the C0 is already giving the right condition. OK? Is that clear to everybody? This is the correction to that, to this. The tree level graph in the amplitude here gave minus IC0, so we already got what we want. When we go to our loop, we just want to make sure it doesn't contribute, and that forces us to put the plus mu there.

And so what this mu is doing is tracking the power divergence that dimreg in MS bar did not see. There was an integral is power law divergence and our cut off, mu r, has appeared in the numerator and has tracked that divergence. So we're doing a standard thing here where we split the coefficient into the bare coefficient and to our more renormalized and countertrend piece. And unlike MS bar in this particular normalization scheme, there's a finite correction there. And we have a renormalization group equation.

And if you work out what that is from the counterterm, it turns out that only this one loop, it's one loop exact. Showing that is a little more work than I've done, but it turns out that the beta function is one loop exact, higher bubbles don't give any contribution to this beta function, and this is the anomalous dimension. So we could just calculate all those bubbles, and pose the same type of thing. And I've given you the reference that does that. So there's a renormalization group equation in this scheme which we didn't have in MS bar. In MS bar it was scale independent. We also have a connection to the MS bar scheme because if mu r was 0, that corresponds to what the MS bar result, right? So C0 of 0 is where you can think of putting your boundary condition, which is matching to the experiment, which is the a.

And the advantage of this offshell subtraction is that we have a mu, and we can go somewhere else. And if you look at the solution, of the RG with that boundary condition, you see something interesting. So this is the result for the coefficient. One over mu r, minus 1 over a. So if mu is of order P, which is much greater than 1 over a, like we want, then the right counting for the C, which is a function of mu, is 1 over M mu. OK? And so what we've done here is we've swapped 1 over the physics scale that we're integrating out for 1 over the scale mu, which we're taking to be of order the physics we're keeping. OK?

And this is relevant now. This is a relevant coupling with that change. We've made it much bigger by just switching to the physical renormalization scheme. And if we take mu of order p in that scheme, this all of a sudden becomes an order 1 effect with leading order in the Lagrangian, because we have effectively P's downstairs in the coupling. OK? So this scheme allows us a way of thinking in the effective theory way of thinking of having a power counting where we'd have to keep the C0 to all orders.

AUDIENCE: I thought it was order-- AP was order 1.

PROFESSOR: Yeah, AP is order 1. So if you look at the Lagrangian, then you have to go through the counting of how many powers of P the nucleon field has. Which you could do in the same way that we did with the mass. And if you do that with the four-nucleon operator with an extra power of P downstairs, you will find it has the same P scaling as the kinetic term. OK? I didn't go through that, but--

AUDIENCE: So that justifies this, even when P is not much bigger than 1 over a? Is that what you're saying?

PROFESSOR: Yeah, so if P is much bigger than 1 over a, or P is even order or 1 over a, which is a sort of also-- it's also fine with this counting. If P is of order 1 over a, these two terms are sort of comparably big, but you could also use this approach for that case, as long as you're not in the case where P is much less than 1 over a. So if you're in that case, then you should really think about expanding this out some sense. And then you're getting back to the end MS bar result, the MS bar result would have been fine. But to get away from the MS bar result, and think about physics, mu of order P, we could use this scheme instead of MS bar, and then we actually see that we get a reasonable power counting.

AUDIENCE: OK, Because my concern is, what about when mu is like, what about this pole that you're going to--

PROFESSOR: Yeah, we'll talk about the pole, yeah. It's coming up. All right. So it's interesting to think about this from our renormalization group point of view, which is kind of what I was doing when I wrote down a beta function that I was-- here I was just after this solution, because from that solution I got the power counting that I want. Which is that the C0 term is the same size as the kinetic term, and both are relevant.

So we can do that just like we counted P's for [INAUDIBLE] theory, or just like we counted-- [? just comment. ?] These guys are all relevant as long as we count this mu as 1 over P. 1 over mu is 1 over P. So there's another way of thinking about this, and thinking about these different cases, and that's just to think about the beta function itself. So if you look at the beta function for the C0 coupling, and we just put in the solution, plug this back into that equation here, with some constant out front, and then it looks like this. 8 times mu, 1 minus a mu, squared.

If we want to talk about all possible values of a, well a can go from minus infinity to plus infinity. So it's useful if we want to draw this to map it to a compact interval. So let's do that. Move the tangent. And let's just plot beta as a function of x. So there's three values, actually where beta vanishes. There's one value where it blows up.

So this is-- here is x equals minus 1. This is x equals 0, this is x equals plus 1. And what it looks like dips down here, goes there, blows up, then it comes back down, it goes like that. So that's what the beta function looks like.

So this point here corresponds if you think of being at fixed mu. Say you're studying the physics at fixed mu. This point corresponds to a equals minus infinity, this point corresponds to a equals zero, and this point corresponds to a equals plus infinity. So there's three points where the beta function vanishes, if you think about a space for fixed mu. Use some color.

So you can ask about nature. So nature told us the value of a, so what does that correspond to? So taking some value of P and mapping it to some value of x, kind of generically the kind of point that you're interested in is here. This is a1S0, sits there. And then a for the 3S1 makes it kind of here.

So what's going on in this case is that you're not near this fixed point, you're actually close to this one. And you're not near this fixed point, well actually there's an infinity in between, you're closer to this one. This size you should think of as 8 MeV, generically, and then this is like the 8 but there's also a pole in between. So three fixed points.

When we do perturbation theory and we expand about fixed points, and one way of saying what was wrong with dimensional analysis was it was just expanding about the wrong fixed point. The pink one. So a equals zero, was the non-interactive one where we just had the relevant interaction being the kinetic term, but none of the interaction terms are relevant. And a equals plus or minus infinity are interacting fixed points, where you have an interaction that is relevant.

So you can think about that in the following sense. Classically, what a is measuring is kind of the interaction size, if you have classical scattering across sections 4 pi a squared. And if a is small-- if a is either very small or very big, then basically it's the same on all scales because it's either infinity or 0 and it looks the same to particles of all different momenta. OK? And that's one way of thinking about these fixed points. That the physics can't-- the physics-- yeah. Just what I said.

So what about this infinity? That's also interesting. So when a is one over mu, beta goes to infinity. And this actually-- there is a reflection of this in the theory, it corresponds to a bound state. Which I think we'll talk about next time, but. So there's a bound state in the theory, and actually if you start from this side, you can never see that bound because you're doing perturbation theory. You don't see perturbation theory and bound states.

If you start from this side, the bound state is actually just in the theory. We'll talk about that next time. And so it's a state in the theory, you can go and you could find the pole in your amplitude, it's just there. Corresponds to a physical state of the spectrum, and it's called the deuteron. So we have a non-interactive theory, we have some non-trivial amplitude, and this deuteron is a pole in that amplitude.

And you never see a pole if you use the perturbation theory. And if you actually look at the energy, the binding energy of this state, it's also small. Characteristically small, and it's actually related to the fact that the binding energy is small just related to sort of the natural size of this a. We'll talk about that next time.

So what I was saying before about the theory at all scales looking the same, when you have a equals 0 of course it's not interacting. Or when a is equal to plus minus infinity, looks the same at all scales. That means that scale invariant. So it's a scale invariant theory when a is at these fixed points.

And something I worked on was the fact that these points are actually conformal fixed points. So that-- there's a conformal symmetry of the theory that exists at those points, we'll talk about that a little bit. There's another symmetry, too, which I have to mention. And not only because I also worked on this one, because it's good to enumerate all the symmetries.

There's actually a combined spin, isospin symmetry, that turns into an Su4 in this limit. So much like in heavy quark effective theory where the mass was big, new symmetries popped up. Same thing happens here, when the scattering lengths go to infinity, and you go over to those fixed points, there's new symmetries that pop up. One's a conformal symmetry and the other one's a spin, isospin symmetry.

All right. So this looks interesting. You could ask the question, did this pick-- did this physical picture that we developed depend on picking this renormalization scheme that I told you about? We kind of gave up on dimreg, we went over to-- well, we gave up on MS bar, we went over to this scheme which was offshell momentum subtraction. In general, people don't like offshell momentum subtraction because it makes calculations more difficult once you go to higher orders. Well here, are the calculations are not so difficult, so we could do them, but-- you might be interested in adding pions to this theory, or coupling external currents like photons, and then the calculations would get more difficult.

And you'd like, for example, to have a dimreg MS bar type description of this power counting rather than a kind of minimal subtraction. Could I get, could I kind of dress up minimal subtraction to get the same physical picture? That's a reasonable question to ask. The answer is you can. So there's something called power divergence subtraction scheme, different scheme than MS bar. So the PDS scheme. And what it says is, don't just subtract poles at D equals 4, like you do in MS bar, which are corresponding to logs at the cut off, but also subtract poles and D equals 3.

And dimreg knows about power law divergences and they're just poles at different places in the dimensions. And so if we subtract poles at D equals three, we can track the power law divergences in that way. And it's the power of divergence that's actually causing, if you want to think of it as a change to the anomalous dimension, where the anomalous dimension was saying this thing was irrelevant, to changing it to something relevant, you need a big change for that to happen. And the big change that's occurring is coming from a power law divergence here. That's what sort of allowed you to jump, if you like, from this fixed point to this one.

The renormalization group, including the power law divergence, allows you to even flow between those points. Usually we think that power law divergences aren't doing anything, here's an example where they are. They're not doing anything as long as you know you're at the right fixed point. If you're describing the right physics around one of these fixed points, you can concentrate on the logs, but if you don't know where you are then the power law divergences could be crucial.

All right, how does this scheme work? So this is a dimreg-type scheme. So we're going to get the power of mu from the mu to the two epsilon that we have out front. So if I just write this guy down in D dimensions, here's what it looks like. And I've normalized mu slightly differently than we usually do just because it's convenient for this scheme to do that.

So it's not exactly the same as MS bar, it's mu over 2 that I'm putting in. Other than that, it's the same kind of set up as MS bar. You'll see why I want to put that 2 there in a minute.

OK, so this is just the result that we would write down for dimensional regularization. Dimensional regularization is not a scheme. A scheme has to do with what we subtract. Dimreg is just how we regulate. So now, look at D equals 4. So in D equals 4, OK. There's a bunch of factors. This is just giving, this is the answer I quoted to you before. Something finite. And if we look at D equals three, then we have a pole because of that gamma function.

And I've put the 2 in here just to cancel that 2. And so, what this scheme says is to add a part subtraction for this guy. So what we do is, we add a counter term. It looks like minus IM over 4 pi, mu, got one power of mu. Over 3 minus D. C0 squared. And then if we take the graph, plus the counterterm and we set D equal four, which is where we actually want to do physics, lo and behold. In this approach, you get actually the same answer as in our offshell momentum subtraction scheme. And that's just really because this scheme tracks the power correction, the power divergence, just like the offshell momentum subtraction did.

So we just invented a dimreg style of looking for poles that can track the same physics, and we just have to look at poles in D equals 3 rather than poles in D equals 4, OK? And this is easier in general than the offshell momentum subtraction scheme, although for basically everything we're talking about today you could do either one.

Now, where is the predictive power of this effective theory? So far, we've just kind of cooked things together to make the C0 do what we want. Well, we didn't completely cook things together. We switched to another scheme and it kind of popped out naturally, but you can say, well, why not explore three other schemes and see if they work? But let's just imagine that we got things to work, as we just did in two different ways by tracking the power law divergence. The predictive power becomes from now the fact that if I say that's my fine tuning, that C0 got enhanced, I can now predict the size of all other operators in the theory. And other operators like C2 and C4, the power counting we assigned to them previously is not true. We have to figure it out. But we can figure that out once we know what approach we should use.

OK, so this is same as above. I won't go through it, but you know, same anomalous dimension, et cetera. And it's easier in general. Let's think about C2 mu. So if you look at C2, the first kind of diagram you could think about would be a guy with one C2, and then this is the first type of loop diagram you might think about. And these guys have a P squared because C2 gave a P squared. So they have an extra P squared. And they diverge.

They also have a power divergence. So if you calculate in either one of these schemes, offshell momentum subtraction, or this PDS scheme, these guys get a divergence. And again, it's a power law divergence so there's a mu here. And you get a beta function, that's that. 2C0 C2.

So if you take the boundary condition C2 of 0 which is our MS bar result. 4 pi over M. A squared, r0. And you solve this, you find C2 of mu is 4 pi over M. 1 over mu minus 1 over a, squared. r0 over 2. So there's two-- we previously, with our counting, when we were counting dimensions, we would have said C0 goes like 1 over lambda, C2 goes like one over lambda cubed. We had 2M plus 1 lambdas. What we've just discovered is that yes, there's a lambda from this r0, that's a 1 over lambda, but the other two lambdas are really mu's. So this operator is also enhanced, and it's enhanced by two powers.

So once you know leading order theory, you should be able to determine the power counting of all of the other operators, especially if they're not relevant ones. You have to get the relevant part right, and then you can use that Lagrangian to predict all the scaling for everything else. And that's what we've just done. Gone to 1 over mu squared lambda. OK? And those powers of mu we see in the scheme, and the 1 over lambda comes from the r0.

OK. So what the RGE actually does is it tells us-- one way of thinking about it is that it tells us the enhancement, due to fine tuning, of all operators in the theory. And that's really because the fine tuning was just a change of our power counting, and we have to propagate that change everywhere. And we can do that, and it's the beta functions that tell us how to propagate the fine tuning.

So if you keep going, you can do C2K of mu. You find an anomalous dimension. This theory is kind of nice, you can basically do all the calculations, so. When you go to CK, you get contributions from various lower order coefficients, and it's one loop exact so you only have pairs. And then you can kind of contrast what's going on in a naive power counting where P is much less than 1 over a.

Let's just go up to C4. Versus this kind of improved power counting, which is valid when PA is greater than our order 1. So C0 hat it went like 1 over mu, no suppression there. C2 hat goes like one over mu squared lambda, and that actually is irrelevant. But it's just irrelevant by one power. So relative to this guy, it's down by a P over lambda. And then interesting things start to happen with the higher ones, at least from an RGE perspective. So these guys start to get more than one term. But this guy actually doesn't introduce a new constant.

There's a piece of the anomalous dimension of this guy that's actually just fixed by the constant that you already had here, and then there's a new piece. So the new piece is down by two powers of lambda. And that's encoding things about the amplitude, actually, but-- OK, so that's just a little table to kind of convince you that once you have a beta function that you can compute for the coefficients, you can quickly propagate this enhancement from the fine tuning to the rest of your theory. I.e., figure out what the power counting is for all the operators.

AUDIENCE: So every time there is a power law divergence, should I be worried if I'm using MS bar, should I be worried that the power counting could be wrong?

PROFESSOR: Every time-- Yeah, every time there's a power law divergence, it's worth thinking about whether it had some physical impact on what you're doing, I think. If you know you're expanding-- if you can convince yourself that you're expanding around the right fixed point then you're OK. That's my equivalence claim. But you don't necessarily know that. So let's go back to our amplitude and see what's going on here, and see what it looks like with this power counting. And so it's really just a different expansion of that amplitude that we had.

And in either the PDS scheme or the power diversion subtraction scheme, we end up with this amplitude in the case where we would use that scheme. So you can see in PDS that if I set this coefficient to 0, the denominator becomes 1, and then I get with the offshell momentum subtraction scheme. In PDS is a little harder see that it gives just that, but it actually gives the same thing in either scheme. And if you think about what type of expansion you're doing here, you're keeping C0 to all orders. So your amplitude at lowest order is just this, and then the C2 term looks like that.

And then there's some higher terms which I wrote in my notes, but I want right here. And what this is, this here is some kind of interaction. I'll make it a bigger circle, which sums up all the bubbles with C0's in them. That's what's happened here. And this here, if you like, is like taking C2 and then dressing it with bubbles on either side. So there's two, there's bubbles. The bubbles on the other side. And then bubbles on both sides.

So we calculate these loop graphs and these are the amplitudes we get, and that's because we're treating the C0 coupling to all orders, we're summing it up. And actually, each of these amplitudes, if you look at the RGE, is mu independent. Explicitly mu independent. So it's like perturbation theory, where we're doing a momentum expansion, and order by order in that momentum expansion, the amplitude is independent of the scale mu. The only purpose of the scale mu is to help us think about power counting of these operators. In the end of the day, when we make physical predictions, then getting mu independent answers. OK. And this is like organizing, if you like. And you put it in terms of-- put it back in terms of a, this is like organizing the theory in this way, where you keep all powers of AP, and that makes it very clear that it's mu independent.

Now, this part of the theory is so simple you could have figured that out just by writing the top line down in terms of a's and P's and just writing this line down. But you could also use what I've been talking about to figure out, for example, say I coupled an external photon to my four nucleon operators. How big is this? OK. Well, it actually gets enhanced from the fact the scattering length is large, and you can figure out how important this effect is, and when people do things like deuteron formation in the sun and stuff like that, they use this effective theory to do higher order calculations and make precision predictions for deuteron physics.

So it's not just a toy model. It's actually something that has a real impact on some physics. We'll talk a little bit more about it next time. We'll talk a little bit about the conformal symmetry and I'll talk a little bit more about the deuterons since that's something interesting in this theory, and then we'll go on from there.

So, any questions? It's cool stuff. Simple to do calculations. It's kind of interesting to think about.