**IAIN STEWART:** All right, so this is where we got to last time. So we were on the road to deriving the leading order SCET Lagrangian. We said that there were some things that we had to deal with, in particular expanding it.

And we started out, as a first step, getting rid of two of the components of the field that we said are not going to be ones that we project onto in the high energy limit. And that led us to this Lagrangian right here, but we haven't yet expanded it. We haven't yet separated the ultra soft and collinear momenta or the ultra soft and collinear gauge fields.

And so in order to do that, I argued that you have to make multipole expansion. And I talked a little bit about what the multipole expansion would look like in position space. And then I said, it's better for us to do it in momentum space.

And we started to do that. And we're going to continue to do that today. So first of all, let's just go to momentum space.

So I've called the field here-- since it's not the final field that we want, I introduced a crummy notation for it, Cn hat. And once we get to the final field, it won't have a hat. And that'll be the one that we keep using from then on.

So let's take this hatted guy, which is not the final guy, Fourier transform him and talk about doing things in momentum space. In momentum space, I'm going to make an analogy with what we did in HQET. In HQET, we split the momentum into two pieces.

We had a piece that was large and a piece that was small. And then we could expand. I'm going to do the same thing here, split it into a large piece and a small piece where I'll call the large pieces the one in the lambda components. And then the small piece will just be all the pieces that are of order lambda squared.

So if you draw a picture for this, it looks as follows. How should I think about what I'm doing here? Here's how I want you to think about it.

So let's draw a two-dimensional picture in minus perp space. In plus space, there's no split because there's no plus here. So it's really in the minus and the perp space that we have to make a split.

Yeah, let's see. All right, so I'm going to draw a grid. And the way I want to make this split precise is I'm going to think-- OK, so I should have maybe made my axes in a different color to distinguish them from the grid.

Let's make them orange. So the way I want to think about these two types of momenta is I'm going to think about these Pl's as discrete and the Pr's as continuous. So let's think about the Pl's as labeling these points at the center of each of these boxes in the grid.

This one was supposed to be-- that's fine. My squares are not the same size, but they roughly should be. And so then we can think about, if we specify some momentum in the space-- let's say we specify momentum that points to this point right here. The way that I can point to that point is I take one vector that points to the center. And then I take another small vector that points to that point. The large vector is the Pl. The small vector is the Pr.

OK, so this guy here is Pl. Maybe I should make the other one a different color, Pr. And so the way that we should think about this is in the following sense.

We should think about the average spacing in between these grid points here is something that gets a power counting that makes it of order Q lambda. And the average spacing between grid points in this direction is something that makes it order 1. But if I ask about distances inside a box, if I ask about a distance like this one, that's of order lambda squared.

So to get between boxes, you need to make a big jump to the larger momentum. But to move around in a box, that's only small momenta.

**AUDIENCE:** So why can't you have order lambda basically inside the box in the P minus dimension?

**IAIN STEWART:** Yeah. So you could. So you can ask, where is in order lambda momentum in the P minus?

**AUDIENCE:** Yeah.

**IAIN STEWART:** And it's in a nether land. Either way, you can think about it as being part of this one or that one. It doesn't matter. Because technically, if I just talked about minus, I'd only be interested in separating a 1 and a lambda squared. And whether I call this lambda squared or I call it lambda or I call it eta-- let eta equal lambda squared-- I'm only trying to separate two things.

So if you talk about some momentum that's in between, you could ask the same question actually about, what about a lambda to the 3/2 momentum in the perp space? And that's the same question you just asked about the minus space, right? And that's something that we don't really care about because there's not going to be momenta that are scaling like that. So we don't really care about separating it.

OK, so Pl are discrete grid points. And Pr-- continuous. And this picture also makes clear that, if you take any P, it uniquely is given by some Pl and some Pr. If I point at any place in this grid, you can tell me exactly which Pl I should pick and then what Pr I should pick.

This is, in some ways, just going to be a tool. And in the end of the day, we can kind of dispense with this picture. But it'll be a useful way of thinking about what's going on and getting to sort of the final results that allow us to dispense with this picture.

So what about integration? So if I integrate over all P and P is collinear, what does that mean? So it corresponds to summing over all the grid points and then integrating over all the residuals except for one location.

And that's the special box here that has label equal 0 because we know that the power counting of the collinear momentum is such that it always has to have a non-0 Pl. And so the one box where Pl is equal to 0, 0, that particular box doesn't have a collinear momentum.

And actually, that box is exactly what you think of as the ultra soft momentum. So if you have an integral d4P and P is ultra soft, there's no sum. And you just have interval d4Pr. And that's like integrating over the 0, 0 location in the grid.

OK, so you can see that together we cover all possible places for an integration. And we actually don't do any double counting with this setup. What am I going to do with my fields?

Well, because of this split into two different pieces, I'm going to say that my fields actually can be labeled by Pl and carry Pr also as an index or as a function of Pr. So in order to make this explicit, we're going to write our Fourier transform guy with a label Pl and an argument Pr like that. So the name label comes from the fact that we do this.

And there will be some analogies with why we called this a label in HQET. We'll see to what extent that analogy carries through in a minute. Another thing you should note from this grid picture is that you have separate momentum conservation in the label and the residual.

So if you think about some integral d4x of something that you split between label pieces and residual pieces, then the right way of thinking about this is that you have a discrete [INAUDIBLE] delta for the label and then a continuous Dirac delta for the continuous variables. And there's some 2 pis.

And so this is what would be meant by-- if you have two momenta that are equal, you know that they're in the same grid square. And then they have the same little green arrow as well. That's what that's saying.

So when we talked about what happens with the multipole expansion in position space, we said that effectively you have a non-conservation of momenta. OK. We had a field at 0. We worked through what that meant. And it meant, basically, that the momenta was conserved on the other line, but not on the one that you were doing the expansion on.

The nice thing about this type of setup is that we have conservation of momenta, but we actually have two conservations of momenta. So we don't give up momentum conservation. It's just that we go over to a picture where we have two of them. So let me show you how that works.

Let's say we have an ultra soft particle, and it's hitting a collinear quark. The collinear quark, because it's collinear, carries two types of momentum. It carries a label P and a residual.

But the ultra soft guy, it's only residual. So let's just call that k residual. There's no label to that ultra soft guy. So if I just look then what this guy's momentum is, it's P label for the large piece because that is not affected by the ultra soft guy. And then it's P residual plus k residual.

OK, so we have a conservation in the residual space, which is the standard one. And we have a conservation in the label space, but the ultra soft particles just don't contribute to that. So we have two conservation laws.

And we just know that certain fields can carry these label momentum and other fields can't. All right, well, since Pr here and kr, the residual components, is just standard quantum field theory-- the only thing that's special is the Pl. For the Pr, what we're going to do for the residual momenta is we're going to transform back to position space.

And that's kind of what we do in HQET as well, right? We treat this guy in momentum space as a label and this guy as a position space for the residual. And the reason we really want to do that is related to locality. And I'll say some more words about that.

**AUDIENCE:** [INAUDIBLE]

**IAIN STEWART:** Yeah.

**AUDIENCE:** I have a question.

**IAIN STEWART:** Sure.

**AUDIENCE:** So this top equation--

**IAIN STEWART:** Yeah.

**AUDIENCE:** So are you bounding the d4Pr by lambda squared, or is there some reason why the integral should fall off before you get there or something?

**IAIN STEWART:** Yeah, we're going to talk about that. So, so far, this is just notation. If you think about it from this grid picture, you'd think some hard walls put in. That's not quite the right way of thinking about it. It's actually not right for several reasons.

And we will come back and talk much more about what the right way of thinking about it is. But the physical picture that you're in this box is the right way of thinking about it. The tricky thing is that these boxes are infinite. Each of these boxes is infinite. And we'll have to see why that is.

In what I've told you so far, you'd think that the d4Pr would really be a box with some fixed edges. And it's going to turn out that the d4Pr-- we're going to use dimensional regularization-- it's going to be an infinite box. But we're still going to be able to do exactly what I'm telling you.

And it's going to make sense. We're going to want infinite boxes to avoid breaking symmetries, like gauge symmetry and Lorentz symmetry. And we'll come back and see how that works.

So in some sense, what I've told you is a Wilsonian story. And we're going to have to make it into a more continuum friendly story. But the moral of the story, the way you think about where these momentum modes live, what kind of momentum they carry, that is just true. So that part of the story, which you can think of so far as Wilsonian, is true. And it's OK to think like that.

So we'll see actually what we do in practice a little later on. Where this will really come in is when we start wanting to think about doing loops, but we won't get there for a little while yet.

So the conservation law in the residual just corresponds to locality. We learned all about locality yesterday in [INAUDIBLE] seminar. And so we can make things local by just going back to the x base for the residual momenta.

OK, so we go back to the x base, which is the Fourier transform conjugate variable to the residual momenta. So the fields we're actually going to use are these ones, where we Fourier transform back in the residual, these guys here.

So it's kind of like HQET, where you think about this part as momentum space and that part in position space. OK, so if you talk about ultra soft interactions-- you can see it from over there-- they conserve the labels. So there's label conservation of a field for ultra soft interactions.

The label of the field here is the same as the label of the field there, so that ultra soft guys do not change labels. So this is Pl. This is Pl. And if this is ultra soft, it's still labeled by Pl.

And the thing that can change is the residual momenta. And that's now encoded in locality in saying these fields all sit at the same point x. On the other hand, the collinear gluons cause us to hop between boxes.

So the collinear gluons are label changing. And that's what's different than HQET. In HQET, we don't really have interactions in the theory in the Lagrangian that do that, but here we do.

So if we have a collinear gluon come in, the collinear gluon, again, carries both labels and residuals if I just draw the labels. And this is some Ql coming in. And this will be Pl plus Ql. And so the label of the outgoing field has changed.

There's one more label. And that's this n, right? We also label these guys by a direction. And that is unchanged also by the collinear interactions.

So it's preserved by both ultra soft and collinear interactions. It can only be changed by hard interactions where you would have some hard production that could produce, say, two guys in two different directions. And those type of interactions are typically only going to occur once in your quantum field theory, kind of like a decay, a weak decay where it occurs once.

And then you do perturbation theory around that. Hard interactions are that way, too. So as far as dynamics is concerned, this label n is something that you can think of as being conserved for the field. It's not going to be changed by the Lagrangian, for example, all right?

OK, so, now, we have kind of what we want. And now, I want to write the action that we had before in terms of this guy and see what it looks like. This split of these guys into a Pl and Pr or Pl and x allows us to expand very easily in momenta.

It's allowing us to do the multipole expansion because we have two things to talk about. And we just can compare them. So let's put the pieces together.

So Cn hat of x is the full Fourier transform of this Cn twiddle. And that can be split for a collinear guy. We can split the measure, and we can split the phase.

And we can write the field in this way. And then we decided that we wanted to use this field. Let me see, [INAUDIBLE] two pis here.

So that means we take this measure, we take this phase, we take this guy, and we write it back in terms of that field. So that leaves, if we want to relate the field that their Lagrangian was in terms of in terms of this field that I told you we're going to use, it's this equation, OK?

All right-- a little bit more formalism. So we're going to actually want to have two different types of derivatives, derivatives that can talk to the momentum conjugate to x and derivatives that give this Pl. So we define two types of derivative operators.

One is just, if you like, a standard derivative in the x base of this guy, acts only on that coordinate. And that derivative scales like lambda squared. So that's residual, picking out talking to residual momenta.

Then we need something that can give us the labels. So we define an operator that does that, which we'll call curly p. It doesn't care about x. And it's just defined to give the Pl mu of the field it acts on.

So there's no plus component to this. There's an order 1 minus component. And then perp is order lambda. I want to say how big that guy is.

So then we have a power counting that tells us how big the p is and how big the partial is. And we can compare them, and we can expand in them, OK? So the bottom line is that we now can just say that, like we had a comparison for the gauge field, we have a comparison for derivatives.

And so this is exactly analogous to what we said before where we had M bar dot A ultra soft is much less than N bar dot A collinear and A perp ultra soft being much less than A perp collinear. And in fact, the way we're setting things up is going to make the gauge symmetry also easy. Because by gauge symmetry, these derivatives and these fields should go together.

They are the same size in the power counting. And these guys should go with those guys. So we're kind of doing this same thing we thought of for fields we're doing for momenta here.

So when we talk about gauge symmetry, this will make gauge symmetry easier, this homogeneity between the derivatives and the fields. So that's one of the powers of thinking like this.

OK, so let's see that this notation is actually kind of friendly. So let me try to convince you of that. It looks kind of daunting, but maybe it's not so daunting.

So let's look at this guy, which was equal to this. So what I can do is I can take this and replace it by e to the minus i label operator dot x, right? But the label operator then doesn't care about the sum. So I can push it through.

So as an operator, it goes through the sum. I have that. And for a lot of purposes, in a sense of suppressing indices, we can just use a simplified notation for this guy where we suppress the sum.

So we were thinking about this as a discrete sum. And we can always write out that sum if we get confused. But we could also adopt a notation where we suppress the sum. And for many manipulations, just like you might suppress some indices in a quantum field theory, that's perfectly fine.

And having this label operator makes it easy to do that. So if you have, for example, field products, say I had two fermions, we try to make the notation work for us as nice as possible. Well, if you have a sum of momenta and then it's exponentiated, if you have a product and it's a product of exponentials, then it just becomes the sum in the exponential.

So this is true if we do it for two fields, right? I'd have the label operator acting on this guy and that guy. And if I were to write out this twice with two sums, then because the product of those two exponentials can be just written as a single exponential, this is true. So this guy acts on both fields.

So in this sense, the notation is friendly. And what this phase factor here is really doing for you, if you think about it, if you think about it in a whole product of fields, it's just when you Fourier transform giving the label conservation, OK? So there's going to be this kind of overall e to the minus i p dot x. And it's just sitting there to conserve momentum in the label space.

So that's its purpose. OK, so there's one more step that we have to do before we start talking about the Lagrangian. And that's this little caveat that I had right at the beginning where I said, let's consider only quarks. So far, we've only considered the quark part, not the antiquark part.

So let's see what we want to do about antiquarks. Technically, the way that the field theory wants to work is that the quarks and the antiquarks kind of want to be different fields, but that's very cumbersome. And so we're not going to do that. And we're going to find, again, sneaky notation to treat them as one field even though they want to be two.

The reason that they want to be two is because, if you look at the phase factor for the quarks in the antiquarks in the kind of standard mode decomposition, it has the opposite sign. So look at the mode expansion in QCD. I can write it in a covariant type notation or with a d4p p if I put in the proper projection.

And then we have the quark part with the e to the minus ip dot x. And then we have antiquark part, our b daggers, and the e to the plus ip dot x, OK? So let me think of that as a psi plus plus a psi minus. And we've so far only talked about psi plus.

So for psi plus of x, we have a sum over Pl not equal to 0 e to the minus i Pl dot x of sort of a Cn Pl plus of x. So this is the case we did. If we were to repeat everything we did for this case, then what we would get for psi minus-- and this is why I say that they want to kind of look like separate fields-- is we'd get a plus i Pl dot x.

And in this notation, if you think about the P0, the P0 is the sum of the P minus and the P plus. So saying that, in this mode decomposition, P0 is greater than 0 for these two equations and when we go over to sort of a collinear scaling, it becomes that Pl minus is greater than 0 in these equations.

OK, so that's the sense in which these are different. With a positive Pl minus, we have an opposite phase factor for these two guys. And in both of these cases, it's true that just as kind of a way of remark, we still have this. That didn't change, remember.

OK, but talking about minus and plus fields would just make our life really difficult. Actually, when we first derived SCET, sometimes you're bone-headed. And we went through everything in a much more detailed way that I'm going to present it to you now, wrote out everything in terms of plus fields and minus fields because of this little issue.

Then we noticed the following thing that I'm going to tell you, that we could just put everything back together with a simple definition. And that's because really the only difference is this sign. So if we want a kind of convenient notation, we can do the following.

So far, if you like, we have this guy with Pl's not equal to 0, but the Pl minus should be greater than 0. So we can make use of kind of negative labels by the following trick. Just define a composite field that looks like this where I put the minuses and the pluses back together.

If I have a positive Pl minus here, then that's giving the pluses. If I have a negative Pl minus here, then that's giving the minuses because of what I just told you. We don't just think about this guy as having plus Pl minus. We allow it to have negative. And we take the negative Pl minuses as a way of encoding what's going on with this guy.

So we have different possibilities for M bar dot Pl. And if you like, what the plus guy is doing is destroying particles. And so let's say this is for the-- if I think about Cn of-- let me make my chart like this.

If I think about what Cn of Pl is doing, it's destroying particles and the antiparticles are created. But the antiparticles are going to be created for the opposite sign of the Pl. And then if I think about what the barred field would be doing with this notation, if the M bar dot Pl is greater than 0, then the particles are created.

So this guy is really just the field we were talking about before if Pl is greater than 0. And if Pl is less than 0, it just repeats the whole story for the antiparticles. OK, so I could write everything out in terms of this plus and minus field. But if I just make this simple definition, I could put everything back together and just work in terms of one field, which kind of does what we're used to.

And it's just a sign convention to remember for this Pl. And that sign convention is also easy to remember because the Pl is really following the fermion flow. If you think about the fermion, then if you think about the physical momentum of that fermion, then the physical momentum follows the fermion in the case of the fermion. And it's opposite to the fermion in the case of the antifermion.

So here's a fermion coming into some vertex. There's a antiparticle coming into the vertex. And the Pl goes this way. It follows the fermion flow, whereas the physical momentum is going this way.

And that's the sign we're just keeping track of. And so the P labels are always following the arrows of the fermion lines. They're always along the fermion number with the conventions we've set up.

OK, so with that little piece of notation, we take into account the antiparticles as well. There's no real additional complications since any way you draw a picture it's completely clear what's an antiparticle and what's a particle. And it just follows your intuition.

And furthermore, because the only difference in the mode decomposition was that sign and we've now accounted for it by just making the labels negative, we have this equation, too, for both particles and antiparticles. So we get the opposite phase here because this guy has the opposite sign.

So there's really no additional complications from adding the antiparticles once we set up this notation. OK, any questions so far? What about collinear gluons?

So the gluon field is Hermitian, the original one. And what does that translate into? Well, it translates into the B's and the A's, right? They're both appearing in the decomposition of the field. There's no B's. There are just A's.

And if you work out, using the type of body decomposition that we're doing and using the type of notation that we're doing, what that means, it means that the star of this guy just flips the sign. So that's what Hermiticity of the original field becomes for this notation.

And again, because of the theta functions that show up in the mode decomposition, you can think of associating ql minus greater than 0 as sort of destruction and then ql minus less than 0 as creation. If you want, although it's not absolutely necessary, you could adopt this convention. And then that would just be putting a particular direction to your gluons. And everything that we said before goes through.

And in particular, if we think of decomposing sort of the hatted version of the fields, then this-- just like before. And we can just also, again as before, if we like, adopt the notation where we suppress the labels if we don't need them.

So the gluons really, in some sense, it's just simpler because we don't have to worry about the particles and the antiparticles because of their Hermiticity. It's sort of they're all the same thing. And for most purposes, there's no sense in even keeping track of this convention.

So that's why I went through it quickly because the convention is always kind of obvious from the context. If you're really talking about production of a real gluon or annihilation of a real gluon, then you can keep track of the signs that way. The fermions are a little different.

So because any starred field is just the negative of this guy, we could always write everything in terms of fields that have no daggers for the gluons. But for the fermions, some of the fields are barred or daggered. OK, so what would be the form of having a general label operator act on some combination of daggered fields and some combination of undaggered fields where the gluons are always undaggered?

Well, it just gives the sum of these momenta. And the way it does it-- so sum of the label of momenta. So I'm thinking of all these denominators as label momenta. I'll maybe not put the l.

And the way that the convention works is that we get a minus sign for the daggered guys. That's our sign convention.

And the reason for that sign convention is effectively that, if you think about e to the minus i curly p dot x on sum 5 p1 and then you take the dagger of that, then it becomes 5 p1 dagger of e top the plus i kind of p dagger dot x. And basically, I'm saying that I want to be able to write that like this.

And that convention is useful. So if I want to act forward on this field, which is a daggered field, then I'm going to take the opposite side. And if I have a daggered p, then it would take the plus sign. And that convention is the useful one for keeping track of momentum conservation because then every field has the same convention. We just have e to the minus i p dot x for every single field.

All right, so if we started with some partial derivative-- I mean some regular derivative. And it was acting on one of our hatted fields. So if I write it out in terms of the labels-- put a Pl on it if you like-- we'd have that.

And so there's two contributions from this derivative. It either hits the phase or it hits the field. And so we can write it like-- if we go over to the label operator, we can write it that this partial derivative is the sum of two derivatives, which then we can take the sum over Pl inside and write it as-- so every derivative for the purposes of this exercise is to convince you of the sanity of the notation.

Every regular derivative acting on one of these hatted fields where we hadn't yet decomposed momenta just becomes a sum of these two derivatives. And that's just like the fact that we split the full momentum into the sum of the label in the residual piece. OK, this is the field version of P mu equals Pl mu plus P residual mu.

OK, but the advantage of this notation is that we can then drop the partials if they're smaller. That's the whole reason for this setup. So now, we can expand because we've split up both the fields and the momentum. So we're able to expand.

So let's go back to the thing we wanted to expand. So we had this guy. And we wanted to expand it. And now, we know how to expand it. We've set up a nice way of doing the multipole expansion by introducing some additional notation.

So because of what I just said and because of what I said last night for the gauge field, we have a very simple decomposition for the covariant derivative. It's the sum of a collinear kind of piece and an ultra soft kind of piece.

So this P mu plus i partial mu is just this. And the split of the gauge field into these two pieces is what we talked about last time. And that split of the gauge field actually led to some things that we didn't talk about that we'll talk about later.

So if we now look at the components that are appearing in here, we can figure out what to keep in each case. And it's pretty simple. Well, for the plus, there's nothing to drop. We just keep everything.

And there's also only one type of derivative, so we just keep this. This is exact. All terms here are order lambda squared.

For the perp, we just keep the collinear parts. So if I write out the other part, just for completeness we can drop those terms relative to these terms. Because these terms here are lambda, and these terms here are order lambda squared. So we drop these guys from the leading order and then, similarly, for the minus.

So here, this guy is order lambda squared. And this guy is order lambda 0. So we drop this guy. OK. So now, just plug those things into our Lagrangian.

And then we get the leading order Lagrangian. All of these phase factors can be encoded in an overall phase factor. We get the full n dot D derivative. But for the collinear parts, we only get these piece, this piece, and the piece over there.

And so you can think of that as a collinear covariant derivative. No. So sort of in the obvious notation, we take the two pieces that are collinear and put them into a collinear D and likewise in the n bar direction.

OK, so this Lagrangian here is the leading order Lagrangian with this additional notation. And if you like, we have this kind of D that has both pieces in it for that n dot direction. And then for the other two, we just have the collinear pieces. And that's the leading order action.

Where we've carried out the multipole expansion, after we introduced notation, it became something trivial. So the power of the notation is that, once you've swallowed it, everything else is simple. If I want to go to higher order, I just work out what these terms are. Then I should also worry about these terms. And we'll do that later on.

OK, so what can we note about this? Well, one thing that we worked out earlier was that this should be of order lambda. And that actually implies that, our new field, we were just pulling out phases. And that doesn't change the overall power counting. So our new field is of order lambda as well.

And likewise, before we sort of had a power counting for the measure, which you can think of in our new notation as a power counting for this thing, and that just scales like lambda to the minus 4 as before. So all the power counting that we were doing even before we sort of figured out what the right field to use, that all carries through. And that's why I did it earlier.

OK, so what's going on here, which I didn't really spell out, but I can spell out now, is since this phase is order 1, the power counting from momenta induces a power counting on the x. And it's kind of the larger momentum that dominates.

So x minus scales like 1 over p plus x plus-- remember in the dot product, it's always plus times minus, a confusing thing about light cone coordinates. And so if you count up the powers of lambda, you get minus 4 of them because you get 2 here, none here, and then 2 there. So that's all going through as before.

And then if you ask, what's the order of the Lagrangian, the Lagrangian density here is order lambda to the 4. Because we have two fields, and then we have a lambda squared here and then a lambda here, a lambda 0 here, and a lambda there.

And our power counting made that explicit. So the whole thing is lambda to the 4. This lambda to the 4 of the Lagrange density cancels the lambda to the minus 4 of the measure.

And you get that, when you integrate, this is lambda to the 0. And that's what we wanted. That was our convention for the leading order term, OK?

So the power counting, nothing really changes. We had a power counting for momenta. We just now have a power counting for all these objects that we're talking about.

What about locality? Well, all the fields in this Lagrangian are at x. So it is local in residual momenta. We said that already when we were drawing Feynman diagrams. And that's just true of the action.

And so what that means is the action is local at the smallest momentum scale in our problem. And we would have been surprised if we'd encountered anything different than that. The action is actually also local at the scale lambda.

Even though we adopted this label convention just to separate the momenta, there's no inverse perp derivatives in this action. And so the perp derivatives appear in the numerator. And so it's actually local at that scale, too.

And so if you like this, we've just written something that's in momentum space. And it's the momentum space version of locality at the scale lambda. And we've hidden all the momentum space so it even looks local.

But if we were to write out the indices again, we would see that it's the momentum space version of locality. And so the only scale that it's non-local at is actually scale that we want to integrate out, the hard scale. And that's due to sort of the presence of this kind of 1 over M bar dot Pl type factor.

So that we already sort of saw earlier when we were talking about building up the Wilson line, right? We said, well, there's some off-shell fields. Let's start integrating them out.

And we got inverse powers of this M bar dot Pl, but that was at the hard scale. So we had no choice but to integrate them out. And you can think about this factor here. If it bothers you, you can think about it in the same way that it's not any worse than what we already did.

OK, let's see if this Lagrangian does what we want. Let's look at propagators. If we have a collinear propagator with a collinear gluon, say, coming in, let's look at this diagram.

These are fields in the n direction. This is some guy that's Pl Pr. And let me just take it to be Pr plus for convenience.

Then if you look at this guy and you ask what that propagator is, it comes out exactly how we want it to come out. So there's nothing that gets dropped in the denominator in particular in the sense that all components of the momentum are showing up.

We're still dropping residuals. But if we're just talking about collinear particles, then the residuals are kind of redundant. The purpose of the residual's momenta is when we're talking about both collinear and ultra soft particles.

So the other case that we need to treat is when we have an ultra soft particle. We could, of course, have both of them at the same time, but let's treat this case. So we have Pl, Pl plus some k, which is purely residual.

And now, this guy, because of the derivatives, these derivatives don't see the k. None of these derivatives see the k. So in the n bar in the perp, we don't see the k.

It would look like that. If you like, if you want to make it kind of look more like the Lagrangian, convince yourself that this is the inverse, just divide by this factor. Divide by this factor, then it puts it under here. And that's exactly what this is doing if you ignore the Dirac structure.

It's not playing a role really for the propagator. And same here, if you take this factor, divide it out, then you got the plus piece which is this derivative. And then you get perp squared over minus.

And that's what happens if you put this down stairs, OK? For linear, all the components show up because we didn't drop anything. For ultra soft, only the plus shows up because it only changes the plus momentum of the collinear field.

So the Lagrangian is smart. It knows how to deal with the fact that the propagator should be different for these two situations. And that was one of the things that we wanted, and we achieved it.

So even though the Lagrangian doesn't know what's going to happen, it's smart enough to deal with any possible thing that could happen and give the right propagator. So we don't have to expand any further. The Lagrangian knows how to give the leading order propagator for any situation that we might be interested in at leading order, which are these two situations.

What about Feynman rules for interactions? So that's also interesting. So if we have an ultra soft gluon, and let's say it has an index mu, there's only one term that comes in because it's only the n dot D that had an ultra soft field.

All the other ultra soft fields got dropped. So there's only n dot A ultra soft in the Lcc0. And so there's only one term.

Things are a little more complicated for the collinear. And this is kind of the price that we pay for setting up the notation that we did the way we did. So this guy here has all terms.

Just by pulling a gluon field out of any one of the terms it can couple to any component of the gluon field just like in QCD, but our Feynman rule is a little more complicated because of the way we set things up. And in some ways, what's happened is that we've made the propagator simpler for the collinear fields at the expense of complicating the interaction.

OK, so we couple to all components-- the perp component, the n component, and the n bar component if this is a collinear field. So these are all collinear fields. So one way of thinking about why that kind of had to happen is the following.

What if we didn't have ultra softs? Imagine that we were going through this whole story, but we just had collinears. Then you wouldn't really need to go through this multipole expansion because there would never be two momentum of different size.

And you could actually, just for the collinears, use QCD. The purpose of having this multipole expansion is really because we want to talk about both of these things at the same time. In fact, if you only had collinears, you just boost to the frame where they're not having this boosted scaling.

It's really relative scalings that the effective theory is trying to encode, relative scaling between collinear and ultra soft or between, say, two collinears going off back to back. That's where the power of the effective theory is, in relative thing. So it's only when you have two things at the same time and you want to describe both of them that you get into the kind of complications we're talking about. If you just have one thing, then things are simple. You could just use QCD.

But if you just had QCD for the collinear sector, you'd know that you have a very simple Feynman rule, igTA gamma mu, and a more complicated propagator because you'd have a P slash. Well, here we don't have any P slash. The propagator, if you write out the spin, is simply n slash over 2.

But the P slash has to come back somewhere. And it's coming back because it gets encoded in the way we set things up in this vertex. So these factors of P slash here are just reproducing P slashes. Then in the full QCD context, it would be in the propagator. You can actually work out, if you start writing down Feynman diagrams, that that's exactly what's going on, OK?

So that's kind of the price that we paid for sort of making a Lagrangian that dealt with both things at the same time and made this part really simple. That part became slightly more complicated. But once we know what I just told you, we can oftentimes, you know, carry out calculations here by making correspondences with the full QCD Lagrangian. Sometimes that's done anyway.

They couple to all four components of the field. Actually, once you start learning how to deal with numerators like this, it's different than what you're used to, but it's just, in some sense, as complicated using this Lagrangian. It's just that you're writing things out in components, which you might not want to do, right?

Why decompose things into components, which we're doing here? Why do that if you don't have sort of something that's projecting onto a component? That's, in some sense, the only complication.

OK. And there's one other thing, of course, which is that the Lagrangian also had terms like this. So it's not simply just one gluon interaction. We could have, for example, two covariant derivatives, two of these perp covariant derivatives, for example. And there's a Feynman rule for that, too, which comes out of the action that we wrote down. OK, questions?

**AUDIENCE:** So I guess this, I mean, the effective theory is reproducing the collinear [INAUDIBLE] divergences of QCD?

**IAIN STEWART:** It is, yeah.

**AUDIENCE:** So in [INAUDIBLE] QCD, the higher divergence cancel with interference [INAUDIBLE]?

**IAIN STEWART:** And that'll happen here, too.

**AUDIENCE:** And you have to loops? Or it's happening like a leading order [INAUDIBLE]?

**IAIN STEWART:** No, you have to look at loops, too. So, yeah, there can be cancellations between real and virtual graphs. And there actually can be such cancellations separately for ultra soft gluons and for collinear gluons.

So we'll talk about what you're talking about a little later on. But there is an important point that I'll emphasize since you ask the question. If you ask about the divergences, the infrared divergences of an ultra soft gluon, so the ultra soft gluon, if we go back to sort of where it was living, it was littered here. Collinear gluon was living here.

And if you ask about divergences, what the divergences are coming from are momenta going to 0 or momentum going collinear. And this guy here that lives kind of here can go both collinear, and it can go soft. So the names collinear and soft, which correspond to the names collinear and soft divergences, are not a one-to-one correspondence.

An ultra soft particle can actually have both collinear and soft divergences. The collinear particle can also actually have both types of divergences. And so the names are associated to where they live in momentum space, not the type of divergences that they're encoding.

If I wanted to do something along the lines of setting up the field theory to just exactly correspond to the divergences, well, first of all, no one's ever figured out how to do that. But you could imagine how it would be different than what I'm doing. Because if I wanted to do that, then somehow my collinear field would sort of be sort of sensitive to angles.

It should be like an angular variable because the divergences are coming from angles going to 0. And it's not an angular variable. It's just a regular field.

So setting things up in this way is kind of how we usually think of an effective field theory. We're just talking about momenta small or big. And that's what we're doing, but it comes in the sense that you don't have a direct correspondence with the divergences.

**AUDIENCE:** So you have more divergences [INAUDIBLE]?

**IAIN STEWART:** The way of saying it is that you have more divergences in a way, but they're the same divergences-- yeah. Yeah. So you don't have more divergences, but really what's happening is-- so if you look at this kind of picture, if you look at the kind of infrared divergences that you could have, so let's regulate them with an off-shellness.

You could have infrared divergences that would be like that. But when you do some diagrams, there's kind of infrared divergences that could look like this, OK? I'm just telling you something that you have to trust me.

And this is coming from an ultra soft scale. P squared over P minus, that's this hyperbola. That's the scale of this hyperbola. And the P squared here is the scale of this hyperbola.

What the effective theory does is, any ultra soft diagram, that'll give all these logs. Any collinear diagram will give all these logs. You could say, well, P minus is a hard scale.

So really the only the IR divergences are just P squared, but the effective theory is kind of dividing it up into these two types of logs. And that's actually going to be useful for us. We'll see that that allows us to do some things that are not so obvious how to do in the full theory.

So, yes, it is kind of splitting and IR divergence in these two different types of terms, but that's actually going to be something that's useful. Because in some sense, there's two physical scales in the infrared. There's a sort of collinear hyperbola and the ultra soft hyperbola. And the effective theory is making that explicit.

Other questions? All questions have been good so far. Keep asking them.

**AUDIENCE:** Could you try using two different lambdas for the ultra soft and the collinear?

**IAIN STEWART:** Yeah. So you could if there was a physical reason why you wanted to. And the thing is that, if the lambdas order each other, then there's no point. If there's a hierarchy, then what that would change would be that the n dot D should get expanded, too. But if the n dot D gets expanded, then kind of what is happening is you're really decoupling everything.

**AUDIENCE:** [INAUDIBLE]

**IAIN STEWART:** Yeah. And so you're kind of losing the dynamical connection that we have so far in this theory. So I think it's not so interesting, but it's worth thinking about. OK. We're going to erase this.

OK. So now, that we have a leading order Lagrangian, it does everything that we want it to do. We can use it to calculate Feynman diagrams. It defines the leading order collinear sector in the effective theory.

There's one more thing I want to do. And that's come back to the fact that I told you before, but now that we can actually make more explicit. And that fact was, when we were talking about currents and integrating that off-shell particles, we saw that you start with these guys. You start attaching them. And you end up with a Wilson line.

I'd now like to make it obvious that that kind of manipulation always is true, that we can always get rid of this field and put it in terms of a Wilson line. And the notations that we've introduced are going to allow us to do that. So let's have a little aside on that fact.

So if you are in momentum space, the Wilson line has an equation of motion, or a defining equation if you like, that's this. This defines the direction, that the Wilson line is long n bar dot n bar direction. And the x and the minus infinity are telling you what the end points are.

So this is the defining equation. You can look back at Peskin if you want to see how that all works. And that's the defining equation in position space.

But we've just advocated for a kind of momentum space notation. So what's the corresponding equation to momentum space? It's pretty simple. If we go over to our momentum space Wilson line, we just go over to our momentum space covariant derivative. And the defining equation is this.

So the notation, in some sense, is pretty simple. We just have to switch objects. If we write out what that derivative is, then it's that, OK?

So now, consider the following thing. Consider this combination, but let me think about it as an operator equation. So let me think about it as if there could be something to the right.

If there's something to the right, than the P bar can act on that thing, too. So I get 0 if the P bar acts on this thing. But if the P bar acts through, then I get just P bar on the operator.

So as an operator equation, I can trace the O. This is true for any O. And we have this operator equation.

But I know something else I know that the Wilson line is at e to the i something. And actually, it's unitary. So W dagger W is 1. And again, in momentum space notation, that's kind of the obvious thing. W dagger W is 1.

And that means I can multiply this equation here on the left or the right by W daggers. And so if I do that, let me multiply it by W dagger. And I can write i n bar dot Dn is Wn P bar Wn dagger.

So the gauge field, g n bar dot An, because of gauge symmetry, it's always going to show up in the covariant derivative. So it's always going to show up in this combination. And this equation is telling me I can switch that for Wilson lines in this P bar if I want to.

I could have also multiplied things the other way around and written it P bar. So sort of by way of completeness, I could have written that as well.

It's also easy to convince yourself of the following. 1 over P bar is Wn dagger 1 over i n bar dot D W. And 1 over i n bar dot D-- I'll check that I got the daggers in the right place.

But these two equations are true, kind of the inverse of those equations. So how do we check that? Well, just multiply this times this. You should get 1, right?

And if I multiply, say, on the right here, then the W dagger here kills the W dagger. Then the P bar kills the P bar. And then the W kills the W dagger. So those are true.

And so we can put this into the Lagrangian if we want. And we can do what I said, that we can actually, if we want to, get rid of the n bar dot A field for the collinears in terms of Wilson lines. And we'll use this from time to time when it's convenient.

If we use the D Lagrangian, the Lagrangian becomes this thing. So that 1 over i n bar dot D becomes this thing. And in some sense, you might think that this is easier to expand if you know the expansion of the Wilson line.

And we just have to let a 1 over P bar act on this full combination of fields. And it just gives the total n bar dot P momentum of that full combination of fields. I claimed this was true. And now, I show you in a kind of non-trivial case how, with these operator manipulations, we could just quickly go back and forth. And it's related to the fact that it appears in this combination.

We'll talk more about gauge symmetry later on. I'm alluding to it from time to time, but we'll talk about it more precisely later on. We have to do this story for the collinear gluon Lagrangian.

And much of this is just repeating what we've said for the quarks for the gluons. And once we know how to write down the covariant derivatives, then it's actually not so complicated. And there's just one little complication that we should be careful about, which I want to emphasize to you. And so we'll go through it.

Let me write the Lagrangian in QCD in terms of traces. And let me use a general covariant gauge. So this is a general covariant gauge fixing term. That should be called the gauge parameter tau. Then we have some ghost fields as well.

And of course, g mu nu is the usual g mu nu A TA. And that's my sign convention commutator of two derivatives with the pi over g. So we do the same steps to get SCET.

And so really, if you think about writing all the g's in terms of D's, we already know how to split the D's in terms of ultra soft and collinear parts. And so that's what we do. Let me write out a D, which is kind of a leading order D.

And we'll just set up some notation. And then I can just basically write down the answer. So let's set up a curly D that has all the sort of pieces that will survive at leading order.

So all the pieces that survived were the collinear pieces and perp in n bar and then, in the n direction, both pieces. And effectively, I'm just changing all the regular Roman D's in my gluon action into these curly D's. And that's almost the entire story.

So from a kind of power counting perspective, once you take into account that, when you dot things, n's are always dotting with the n bars, so it's this times this. So this is lambda squared. This is order lambda 0. This guy dots into himself.

So you get lambda squared there as well. So for power counting purposes, when you start squaring things, they're all going to be homogeneous with this setup. So basically, that's what I'm saying here.

And then there's one complication, and that's this partial derivative. Well, there's two partial derivatives. There's one there, and there's one there.

So this is in the gauge fixing term. These terms are both related to the gauge fixing. And then those terms, actually, I want to make them covariant.

So let me define one more thing. I want to make them covariant, but I want to make them only covariant under the ultra soft part of the action. So here is just pulling out the pieces that would have the ultra soft field making a derivative that I'll call curly D ultra soft.

So one way of thinking about that is that it would be like background field gauge. In background field gauge, you'd make this partial view into the D mu of the background involving the background field. And that would be a background field gauge.

So from the point of view of the ultra soft field being a background field, it's very natural to have a replacement that looks like this. And that's one way of arguing. Really what's going on in the effective theory is that this is general covariant gauge for A n. That's what we want it to be for A n mu.

This should be the gauge fixing term for the collinear gluons. And it shouldn't be breaking any gauge symmetry of ultra soft gluons. So we'd actually like the Lagrangian to both have gauge symmetry for collinear gluons and a separate gauge symmetry for the ultra soft gluons.

Since this guy is supposed to be gauge fixing for the collinear gluons, we don't want it to be doing anything to the ultra soft gluons. And that's accomplished if we make this replacement that i partial goes to i curly D, all right? So we could write out what Lcg0 is, but it's just these two replacements.

Since we're out of time, I'll just say that and, you know, sort of write out the fields and the notation that we developed, OK? So we'll talk more next time. What we'll do next time is we'll talk about symmetries.

We'll finally talk about gauge symmetry. And we'll talk about other symmetries. And in particular, symmetries are going to be important for understanding how much of this story that I've told you really carries through once you start doing loops.

Symmetries protect you from certain loop corrections. Gauge symmetry, for example, connects terms to all orders in perturbation theory. And we'll see how those things kind of work next time.

Really, the story I've told you so far is all tree level. We haven't done any loops. One way of handling what loops can do is by looking at symmetries and saying, once we impose those symmetries, what's the most general possible things that loops could do. And we'll do that next time.