PROFESSOR: All right. So, so far we've recently been talking about examples in SET2, and we're going to continue to do so today. So the example that we did last time was the plan photon form factor. That did not have any soft degrees of freedom. It just had colinear and higher degrees of freedom. So it was a particularly simple example of something we could think of in SET2. We'll start with a slightly more complicated example, this decay, B to D pi, where we have both colinear and soft degrees of freedom. This was an example that we mentioned at the very beginning of our discussion of SET, and now we're going to see how factorization looks for it.

And then we'll talk about something called the rapidity renormalization group, which has to do with situations in SET2 where the separation of degrees of freedom is a little more complicated than in the previous examples. And we'll see that there can be a new type of divergence that shows up. And that new type of divergence leads to a new type of renormalization group.

So B to D pie. So there's going to be, in some sense, three hard scales of this problem. The mass of the B quark and the mass of the charm quark are going to be taken to be much greater than lambda QCD, so we'll have an HQET type description of the B quark and the charm quark. And also the energy of the pion, which is in some sense, related to the difference of the B quark and the charm quark mass, will also take that to be much greater than the lambda QCD. So just by kinematics this thing is proportional to MB minus MC roughly. You could say, it's the difference of the squares of the hadron masses.

OK. So let's first-- we know how to treat this decay if we were integrating out the W. This is a weak decay, so B is changing to charm. Integrate of the W boson, run down to the scale MB, which is the larger scale here, that's the electroweak Hamiltonian. So that's what we'll call the QCD operators, which are the relevant description at the scale of order MB. Some pre-factor.

And I'll write the operators in the following way-- a slightly different basis than we used previously, or a long time ago when we were talking about this particular case. So just a different color basis, singlet and octet. OK, so that's our description, where p left is projecting us onto, the left-handed components. So what we want to do is we want to factorize the amplitude. This is an exclusive process where we make a transition between specific states. So we'd like to separate scales in the D pi, and then we have O0 or O8 in this matrix element. So we have two matrix elements, one with O0 and one with O8.

And so, what could it possibly look like? Well, we already talked about the degrees of freedom here. The D is going to be soft and the B is going to be soft. So this is soft, this is soft. This is going to be colinear. And so, if it's going to factorize, and the soft degrees of freedom are not going to talk to the colinear, the kind of thing that you would expect to show at leading order in the lambda expansion is that you have the following kind of process. Let me reclaim this space.

So here's a heavy quark. Here's one of these operators. Here's the valence quarks in the pion. Another heavy quark-- this was B charm, U and D. And there's an anti-quark, and we have to address this by gluons. And if it's going to factorize, then the way that we should dress it by gluons is as follows. We would have soft gluons here, and they could interact, if you like, between things in the B and the D, because the B and the D are both soft. We can also have back and polarization diagrams.

And that is going to factorize from things in the pion which are going to be colinear. So we have our colinear gluons and colinear quarks inside here, and maybe there's some Wilson lines too. So we would expect some kind of picture like that. And that's actually going to be what we do find. But exactly what happens at this vertex, what kind of convolutions there are, that we have to work out.

All right. So what factorization in this context means is that there's no gluons that are directly exchanged between the B to D part and the pion part. So that it effectively factorizes into a matrix element that's like a B to D transition and a vacuum to pion transition. So you can even guess what kind of objects this would depend on. If you have something like this green thing, that's a B to D form factor. So we expect a B to D form factor.

And for the pion, if you have something like this, well, we already talked about something like this when we were talking about gamma star, gamma to pi zero. So for the pion, we expect the pi zero-- the light cone distribution, which is the sort of leading order operator for the pion. So we'd expect a 5pi of x. And we'll see that we do indeed find that.

So the B and D have P squared of order lambda QCD squared for their constituents. The pion is colinear, and its constituents have P squared of order lambda QCD2, but they're boosted. We can again use SET-- this is SET2, but we can use SET1 as an intermediate step, just like we did-- we talked about last time.

So let's do that again. So match-- step one was to match QCD onto the SCET1. So there was some hard scale. And the harder scale in this case could be any one of these three. So collectively I just denote them by Q. And so what's going to happen in that matching is we take these operators O0 and O8, and we have to match onto SET operators. So let me call the SET operators Q0.

I just there's-- because of the fact that there's a heavy quark in the way that that works, there's two possible spin structures here. But we'll see that actually only one of them has the quantum numbers in the end. So we have heavy quark fields, charm, and bottom, that we can imagine that type of operator. And then the colinear part, we can dress it with the Wilson line, as always. And let me put the flavor upstairs. And let me put in the most general Wilson coefficient that I can think of for this process.

This could also depend on V dot V prime. I didn't denote that. But in general, it will. And it could depend on any of the scales Q. So it can depend on the large momentum of the colinear fields, as always. And there's one combination that's not restricted by momentum conservation. So there's one combination that's not related to these scales, and that's the, in my notation, the P bar plus operator.

And then there's likewise, there's another thing with the TA. So same thing, TA, everything the same, TA. And then this has got a different coefficient like that. And so, what are the Dirac structures just to be explicit? The heavy one-- so the light one is going to be M bar slash over 4 in my notation, the 1 minus gamma 5.

And for the heavy one, you could have either 1 or gamma 5. And that's because you originally have a left handed-- originally here you have a left handed guy between the charm and the B. But remember that a mass term connects left and right. So after you integrate out the mass of these quarks, you don't know whether-- you don't know the chirality anymore in this. So that's why you can have both the possibilities 1 and gamma 5.

So if you put any other Dirac structure-- so here you could use chirality. And so, you know that these guys should be left handed still, and that's why that this should be this structure. You know it should be an M bar slash, because any other structure that you would stick in here between them would give you something that's power suppressed. Because you know that N slash, when CN 0, and also CN bar gamma [? per mu ?] with any kind of P left or whatever, is also 0. So you'd have to have something more complicated.

All right, so there's those operators. And when you do this matching, it is non-trivial in the sense that these two operators-- it's not diagonal. It's not like O0 goes to Q0, and O8 goes to Q8. As soon as you start adding loop corrections these two mix, and then they give you some contributions to these coefficients C0 and C8. OK?

So what you mean by octet operator in the electroweak Hamiltonian is different than what you mean by octet operator in the SET1 factorized result. But that's just a complication that you deal with when you are doing the matching. This guy can be proportional to the Wilson coefficient C0F and C08, and that's not really a big deal. Any questions so far?

STUDENT: [INAUDIBLE]

PROFESSOR: Yeah

STUDENT: So is the point of matching to get one of them as being [INAUDIBLE] to get the softs?

PROFESSOR: Yeah.

STUDENT: So you're going to distribute them--

PROFESSOR: That's right. I'm going to do that right now.

STUDENT: OK, so that's--

PROFESSOR: Yeah. So then step two, field redefinition in the SCET1. And let me not put superscript zeros, let me just make it as a replacement. So then we get RQ0 again. And OK. For this guy, it's exactly the same, because in the Q01 comma 5, all the Y's cancel. In the octet guy, it's not quite that way, because there is-- because we do have-- in that case, we do have the Wilson and lines getting trapped by the TA. So let me write out that case. So we have-- so that's what we would get after the field redefinition for two operators.

OK. So the next thing to do would be to instead of calling them Y's call them S's. So, but there's one more thing I can do too. So these are, remember, these are soft fields and these are colinear fields. So this isn't factorized, because we have contractions between these Y's in the fields over here. Gluons can attach to heavy quarks. So in order to factorize we want to move those Y's from there over to here. And we can do that.

So here's how that works. This is a formula that I could have told you earlier. So if you have a Y that lies around a TA, that's just actually the adjoint Wilson line. So this is a formula that relates fundamental Wilson lines and an adjoint Wilson line. So in the adjoint Wilson line, you'd build it out of matrices. If you like, they're like this. So the matrix indices would be B and C, and instead of having fundamental indices for the TA alpha beta, you have an FABC, and this is the kind of thing that you would exponentiate. But other than that it's the same thing. And there's just a color identity relating them.

So because of this identity, you can also write down another identity, which is Y dagger TAY. This guy is-- so Y dagger TAY is just the other YAB-- this guy's an orthogonal matrix, TB. So if you reverse the indices then you get the opposite way. And also, this guy is-- remember this is a matrix just in the AB space. So if I use this formula in here, that allows me to take these Wilson lines here and move them over here. Right? Because I can take them, write them as a Y, and then the Y is just something that doesn't care about-- it just moves over because it's contracted with that index A.

I guess I've got some problems with capitals and small letters. Let's make them all capital. So then not going to move it over here. And then I can convert it back to a Y dagger Y if I want to. OK. So we can take this guy and this thing and write it over there as HV Y TAY dagger HV prime. I was careful about that. This was the prime. And then all the soft gluons-- all the ultra soft ones are over here in this matrix element and all the colinear fields are over there.

So if you like, you could say, we get this, and then we get our colinear matrix element that has TA, but it has now no ultra softs. And so we have a product of colinear and ultra soft things tied together by one index A. OK, so that's just a little color rearrangement. It's useful, because now they're really factorized. And now when you take matrix elements, the matrix elements will factorize.

Oh, sorry, before we take matrix elements, let's switch to SCET 2. So this is, again, an example where it's trivial. Because what we have is, we have one type of operator that we're considering, this weak transition, and we don't have a time limited product of any type of two operators in SET2. We just have a single operator that has both types of fields, and then we have the Lagrangians So this is, again, simple. So we have one mixed operator plus L0 colinear and L0 soft. And those things are already decoupled, and so this is simple.

And so, we simply replace Y's by S, renaming it soft instead of ultra soft. Really nothing is changing. And these colinears, we just put them down onto SET2 colinears from SET1 colinears. OK, so to make it look like I've done something, I'll write it out again. But there's really nothing happening except that now the fields are in SET2, with the correct SET2 scaling.

So there's no Wilson coefficient that's generated by this stuff-- there's no additional Wilson coefficient because of that fact. So these are SCET2 now. And similarly for the octet. So here we would really have the-- OK.

So now we can take matrix elements.

STUDENT: [INAUDIBLE]

PROFESSOR: Yeah?

STUDENT: What do the coefficients look like when they're not just 1?

PROFESSOR: So they would be functions of say, plus times minus momenta. So we could have-- if it wasn't 1, what would happen is effectively-- so yeah, we talked a little bit about this last time, but let me remind you. If it wasn't 1, that would happen in a situation where you had something like this. Some off shell field, O, say like this. So this is a T product of two things rather than just one thing that mix off the colinear.

And this field here was off shell in a way that basically, this field here is an off shell field that would be a product of the plus and minus momentum of these guys. And so, you could get something that's effectively living at this hard colinear scale below the hard scale in the problem, from sort of T products of soft and colinear operators. This is getting a little sketchy, but-- since here we only have one operator, that couldn't happen, because you just start attaching softs. And colinear is been-- it's already factored. So there's no way that you could sort of get this intermediate off shell guy.

STUDENT: [INAUDIBLE]

PROFESSOR: So this guy would be colinear, right? And the way to think about this is like this. This is just a diagram that exists. But then you go over to the SET2, and then you have your change of colinear to the right colinear. But this guy doesn't change. He's still hard colinear.

STUDENT: OK, so it's a matching when the material actually has a different scale than matching--

PROFESSOR: Yeah. This scale would show actually up. And so if it doesn't show up-- so this would be kind of a situation, an going from step one. And actually, we could have done some examples where that happens. But I'm choosing to do this rapidity renormalization group instead. Basically this happens if you look at matrix elements, and you could look at matrix elements where you have some subleading interactions.

And there are examples in exclusive decays where you could have this happen. One example is if you look at B0 to D0, pi 0, just having all neutral charges, then actually this will happen. It'll be-- and it'll be more complicated than what I'm telling you. But you can derive a factorization theorem for this. It's power suppressed relative to the one we're talking about, because the one we're talking about always has a charge pion, and it turns out that that happens at leading order, whereas the neutral pion process with the neutral B and neutral D is something that's power suppressed.

Hope I'm remembering that right. Yeah. I am. All right. So number four, this is an aside. Number four, take matrix elements, and here we find actually that one of the matrix elements is just 0, the one with the octet. So let me write the nonzero ones first. I wasn't too careful about the two different-- about this.

So there's some guy that's just giving a convolution between the Wilson coefficient and 5 pi. And then this guy, which is some normalization factor times a form factor. These things are all mu dependent in general. And this thing here is the Isgur-Wise function, which is the HQT form factor.

And W0 is kind of the kinematic variable that that form factor can depend on, which is V dot V prime, the labels on the fields, and that encodes the momentum transfer. Which here is just related to the kinematics. So W0 is some function of MB and MC, which I'm not going to bother writing down.

STUDENT: [INAUDIBLE]

PROFESSOR: Just some constant. I mean, it could have some kinematic factors that makes up the dimension, like some-- depends on my gamma. Yeah, it's just some number.

STUDENT: [INAUDIBLE]

PROFESSOR: OK so these are the singlet operators. So in the singlet case, we have initial state and final state. In all cases we have initial state and final states, which are color singlets. And these operators are color singlets. In the case of the octet, we also have color singlet states. And we factorize such that we do have a matrix element for example. So let me just write one of them, and no. And this is 0, because there's nothing that could carry the index A in this matrix element.

So the octet matrix element's 0. It's important that we factorized it for that to be true, right? If we had D pi, then we'd have a color singlet operator here, B. So we wouldn't have been able to make this statement in the original operator in the electroweak Hamiltonian. This would just not be true. But once we factored it and put all the ultra soft fields here, that everything that's going to be contracted together, then we can make this statement. We couldn't even really make this statement when the Y's were on the other side. We had to move them over here to ensure that this statement is completely true.

OK. So color octet operator is color singlet states. OK, so then you just put things together and we can multiply these two things to get the final result. So if we write it as a matching from the electroweak Hamiltonian, there are some normalization factors. So grouping together these factors of F pi, E pi, and M prime, which I'm not worrying so much about, there's a Isgur-Wise function, and then there's a single convolutional between the hard coefficient, which is kind of like our example of the photon pi on form factor.

And then the slight [INAUDIBLE] distribution. So we see an example where it showed up in a totally different type of process from the one we were considering previously, thereby showing us kind of the universality of that function. And then there would be some power corrections to this whole thing that we're neglecting, that go like lambda QCD over those hard scales.

OK, so this is the Isgur-Wise function. And actually I did write down what the W would be. So this W would be that, some-- you can write it in terms of the meson masses like that, so it's the Isgur-Wise function at max recoil.

And this function is measured, for example, in a semi-leptonic transition. So you can imagine that the pion distribution function, or properties of it, were measured in the photon pion transition. Is Isgur-Wise function is measured in the B to DL new transition. And then you can make predictions for this B to D pi. OK, so that gives you an example of how you would use these factorization theorems.

So this applies to basically-- this type of factorization that we just talked about, it applies to a lot of different things with charged pi minuses. Or you can make the pi minus a rho minus, and that wouldn't really change anything. So you could have B0. If you wanted to look at charges, you could have B0 to D plus pi minus. Or you could have B minus to D0 pi minus.

And there's a third one, which is the one we were talking about over here, B0 to D0 pi 0. So there's three different ways-- three different B to D pi transitions depending on the charges. And what we've derived applies to charge pions or charge rhos. The neutral ones end up being power suppressed.

And you can see that kind from our discussion there was kind of never a-- there was-- it just writing down the leading order operators, well, maybe you have to work a little harder. But effectively, the leading order operators don't make this transition with the two charges being the same such that you could get a pi 0, so you have to do something more in order to get that case. All right. So questions?

STUDENT: Can you [INAUDIBLE]?

PROFESSOR: You can if you want. Oh, MB over MC. So MB and MC, we're treating both of them as avoiding the same in what we've done. So if we wanted to some logs of MB over MC, we'd have to do something a little different than what we did. You'd have to first integrate [? O to MV, ?] treat the charm quark as a light quark. You could do that.

STUDENT: Would it make the SCET [INAUDIBLE]--

PROFESSOR: It turns out that--

STUDENT: --analysis different?

PROFESSOR: --yeah. So those are single logs, actually, they're not double logs. And it's related to the fact that you have massive particles. When you have massive particles you don't get the extra singularity. And people in HQT worried about something-- logs of MB over MC for a while-- and then after a flight of doing enough calculations they realized it was totally irrelevant, and you should just not bother.

You should just calculate the alpha S corrections, treating MB and MC as comparable, and summing the logs-- if you sort of think of leading log as being more important than the order FS calculation, that misleads you. Sometimes the sign is even wrong. And so there's sort of a general experience that something logs of MB over MC and HQT is not even--

STUDENT: Just because they're single logs?

PROFESSOR: Just because they're single logs. I mean, that's one thing that makes it different than, say, the double logs that you would resum in this process. Actually, these double logs are also single logs. So you could decide whether or not to resum of them. But either-- whether or not you do resummation, this is still useful, because you could make a prediction for this decay rate, and it works really well. All right. You can actually also make predictions for these decay rates. You can predict actually the relations between the D0 and D star 0 using the factorization-- subleading factorization theorem.

OK, so let's move on to our second topic, which will take the rest of today. And that's rapidity divergences. OK. So when we were talking about SET1, and we were talking about loop calculations, we saw that there was a subtlety where when we were doing our colinear loops, that could double count the ultra soft loop, if you remember.

So kind of schematically, I could say, that this true CN was sort of a CN naive minus a CN 0 bin. So you could do a calculation ignoring that, but then you have to be careful, and there's a subtraction. And that subtraction avoids the double counting with the ultra soft. So if you think about there being ultra soft amplitudes and colinear amplitudes, this avoids a double counting.

Now, we never talked about whether there's something analogous to that in SET2, and we just did a lot of SET2 examples without ever even saying those words. So why it's actually-- for what we've talked about so far-- OK to ignore this issue. But in general, it's not OK. So if we go back to our picture of the degrees of freedom in SET2, have this hyperbola, and you could have softs, you could have some colinears. And then the example that we just did, it's like these were kind of the relevant modes. And in general, you might have some guy down here as well. So these are the degrees of freedom in the SET.

And effectively what's happened is, if you want to think about double counting, you're sliding down this hyperbola so this hyperbola is kind of at a constant invariant mass, say, lambda QCD squared, or it could be lambda QCD. And unlike the case in SET1, where this guy lived in a different hyperbola here, to get between them you would be sliding down the hyperbola at fixed invariant mass. So that's a little different.

But in general in SCET2, there are also 0 bins. So in general, you would have something like, met me denote it this way, CN minus CN soft. And what I mean by this, is this is my original amplitude, where P mu was scaling like Q lambda squared 1 lambda. This was the original. And this would be a subtraction where you take that amplitude, and you'd make it scale like in the soft regime. So it would be P mu, lambda, lambda, lambda. So that's different than the example of SET1. In

The SET1 case, we would really just be scaling down the 1 in the lambda, so that they would be both of order lambda squared. Here we're actually scaling down the 1 and scaling up the lambda squared to a lambda. And that's the right thing to do to go from here to here. So we're just taking the amplitudes in this region and subtracting them in this region. And in general, we do have that.

But actually that's not the real complication that shows up in the SET2. The real complication has to do with whether there's any divergences associated to that. If this amplitude here didn't have any divergences, it wasn't kind of log singular, then you wouldn't really care about doing this subtractions, because then there would be no infrared singularities that you're double counting and it would just be effectively a constant. And the constants are always ambiguous.

So whatever mistake you make in constants here you just make up by changing your hard matching. So you don't have to worry if when the colinear goes down into the soft region there's no divergences. And that is actually what's happened in all the examples we've treated so far. That when the colinear goes into some region where it's not supposed to have singularities, that you just end up with no singularities. There's no log singularities.

OK, so, so far there's no log singularities from the overlapped regions. But that's not in general the situation. And we'll do an example in a minute where there are singularities that overlap. And the true difficulty here is the following. If you think about what's separating these modes, you might draw lines like this. Just to draw some straight lines separating the modes.

And remember that we're plotting here in the P minus, P plus plain. And that fixed P squared is like fixed product of P minus P plus. So P squared is P plus P minus, up to the P perp squared piece, which we're ignoring. So you can think of these lines as lines of constant P plus over P minus. And if I-- this is the-- so something orthogonal to P squared. All right. So that would be one way of thinking about-- so you need something that's orthogonal to P squared in order to distinguish these modes.

And the real issue with that is related to the regulators. When you use dimensional regularization it turns out the dimensional regularization is not sufficient to regulate a divergence that would happen when the CN comes down on top of the S. And the reason is, because dimensional regularization regulates P squared. It regulates-- remember, it's a Lorentz invariant regulator. So it's regulating Lorentz invariant things like P squared, not something like the rapidity, which is this P plus over P minus that you would need to distinguish these modes.

So invariant mass does not distinguish the low energy modes. So rapidity, you could define-- is usually defined this way. So exponent of 2Y, where Y is the rapidity, is P minus over P plus. And if you look at the scaling of that, that scaling either lambda minus 2, lambda 0, or lambda squared for the different cases for CN, S, and CN bar. So it's this variable that's really distinguishing the different modes. OK, and that's these lines-- these orange lines are just putting dividing lines between these in rapidity. All right. OK.

So there's a complication that dimensional regularization doesn't suffice. So you can think of it as regulating P Euclidean squared once you do the Wick rotation, for example. So it regulates-- it separates hyperbolas, but it does not separate modes along a hyperbola. It's a way of regulating singularities between hyperbolas, but not along hyperbola. So that's one complication. We'll need an additional regulator. And we'll see that that regulator will eventually lead to a new type of a normalization group flow, which is flow along a hyperbola. It's not a flow in invariant mass, but a flow in rapidity.

OK. So let's explore what can happen in an example where there are these divergences in sort of the simplest possible example. So there's enough going on that we want to make our lives as simple as possible. So what I'll talk about is something called the massive Sudakov form factor. So you should think of it set up as follows. We're going to consider a form factor, and it's going to be a space-like form factor.

So it's a space-like quark-quark form factor. Q of the photon here is space-like. And we're going to think about, rather than having photons or gluons, we're going to think about massive gauge bosons. So this is going to be some kind of Z, if you like. It could be a Z boson. And I'll just call the mass M. OK. So the thing that I'm going to want to iterate is the mass-- rather than doing QCD, I'm doing electroweak corrections-- electroweak corrections from a massive gauge boson.

So this is relevant if we have electroweak corrections in a situation where Q squared is much greater than at M squared. And we're not going to be having gluons. Instead we'll be talking about these massive gauge bosons-- multiple Z bosons, if you like. We could also put the W's in, but let's make it simple and just talk about Z's.

OK, so let's do this example. So in the full theory, you would start with a vector current, say, and you'd want to match that onto SET. Before we do that, let's just-- let me just write down a kind of full theory object using Lorentz invariance. So you could think about the quark form factor. And four massive gauge bosons, this is just some form factor that you can calculate that's a function of Q squared and M squared, and then there's some spinners.

And so, really kind of the dependence is encoded in this F, which is a function of Q squared and M squared. M squared is acting kind of like an infrared regulator. So this is Z boson, there's not a soft singularity associated to it. And so, what you'd like to do-- and in this process-- is factorize Q squared and M squared, i.e. expand this thing in Q squared and M squared, and maybe some logs of Q squared and M squared. So we want to factorize some logs, et cetera.

OK, so what are the type of degrees of freedom that we could have here? So lambda is going to be M over Q, massive Z boson over the energy scale of the collision, of the gamma star. And if you just look at the Z boson, then it could be colinear or it could be soft. So it could be actually three different possibilities. And I could think about doing this effectively in a bright frame. Just like we did earlier. And then what you would have is that the colinear guy is like the quark, and then the anti-colinear guys is like the outgoing quark. [INAUDIBLE].

So you'd be making a transition in this diagram from N colinear of objects to N bar colinear of objects. And you could likewise have a Z boson which could be colinear. Or you could have a Z boson that's soft. And you have a soft rather than ultra soft because of the mass. If Q times lambda is M, and so if we want a propagator that's like P squared minus M squared, P squared better be of order M squared. That happens for softs, not for ultra softs. So that's why we have softs.

And the same thing for colinears, P squared of order M squared for these colinear. So it's really an SET2 type situation, where the hyperbola is just set by P squared of order M squared. We have our mode sitting on that hyperbola. OK, so it's exactly of this type over there. And the thing that's new in this example is that we're going to encounter these rapidity divergences.

STUDENT: You mentioned there's the only form for vertical in your theories, is that?

PROFESSOR: Yeah. Make things as simple as possible. So you could talk about mixed sort of electroweak in QCD, but yeah, we don't. Let's make it-- in some sense that's kind of just like mixing a problem that we already would know how to deal with, with this one. So let's just deal with this one. All right. So in terms of the external court momenta, we can therefore kind of treat them as follows. Let's just let them be of a large component. They're massless particles. So this is P, and this is PR. And these guys are massless-- should have said that.

And if you go through the kinematics, Q squared, which is minus P, minus P prime squared. If you square that, you just find it to P minus P plus. And we're just effectively, if we pick the bright frame-- which is what we're going to do-- each one of those is separately keep going at prime. Call it bar. Each one of those is separately Q. The large-- these guys are just fixed-- both would be Q. All right.

So we could factor is this current with these degrees of freedom. The quarks are colinear. So at lowest order they've just become a CN and a CN bar. And then we have to address that with Wilson lines. And we know how to do that. So let me just break down the answer. We could again follow our procedure of going through SCET1, but it's now so familiar we just know what to write down. OK. So the current would look like that. And I'm not to worry too much about the Dirac structure. So I won't worry, for example, about gamma fives. And we could put that in, it's easy.

So that's the leading order current, and then we'd have leading order Lagrangians, and we need to start calculating. And if we calculate what you would expect from a major from that is, you'd expect that F of Q squared and M squared is going to split up-- given the degrees of freedom we have into some kind of hard function-- and then some kind of amplitude for the colinear parts, and then some kind of amplitude for the soft part. OK, so you'd expect some hard times colinear factorization of the form factor. And this is what we'll be after. So it's always good to sort of have an idea where you're going. And that's where we're going.

So let's consider just one loop diagrams. And it suffices, in order to make the point, just to consider the most singular one. So I'm going to consider-- so there's various loop intervals that you could have to do when you're doing the diagrams if they're fermions. Let's just take the simplest, which is a scalar loop interval. And I'm going to contrast how that scalar loop interval would look if you were doing the full theory calculation with how it would look in the effective theory. And then we'll see where the divergences come from.

So you can think about this as kind of-- the piece where the numerator is independent of the loop momenta, so the numerator just factors out. OK, so if I took our vertex triangle diagram over there, then a piece of it-- where the numerator is trivial and factors out-- would look like this interval. So let's just study this guy. If we did this interval in the full theory, this would be both UV and IR finite. So this is just giving us some result that involves logs of Q squared over M squared.

So it does have double logs of Q squared over M squared, and single logs of Q squared over M squared. But it's perfectly-- there's no 1 over epsilons. So now let's see what-- let's think about what would happen in the effective theory. So we have a kind of analogous loop interval for colinear, where the gauge boson is colinear. There's some numerator that again I'm not going to worry about. If this numerator is constant, it doesn't-- it's effectively the same constant. In the case of the-- if you take the leading order numerator in the full theory, it'll be the leading order numerator the effective theory as well. But the denominators do change.

And so, if we took the N colinear, then yeah, so this guy doesn't change. Because that's just like saying P minus and K are-- P minus and K minus are the same size. But this guy does change. OK, so this guy would be K minus P bar plus, because K minus is big and B plus is big. Both of these are big. So both of those are big in this diagram. And so that's effectively the Wilson line diagram. OK, where the propagator here was off shell, got integrated out, and just became iconal. And the K squared is smaller, so we don't keep it in a leading order term.

And then analogously, for IN bar, it's the other way. So both of these are big, and this one remains. And then they're soft. And in the soft case, what happens is that both of the propagators end up being iconal. And in our SET operator, that's a diagram where we have our colinear lines, and then we have kind of a self contraction of the S.

But we're taking an SN with an SN-- we have a contraction that's like this. We have two of the Wilson lines-- soft Wilson lines that are sitting in that operator. That's a non-zero contraction. That would lead to a diagram like-- that would lead to this amplitude. All right. I'm going to leave a little space here, because I'm going to add something in a minute.

All right. So how do we see that there's a problem with these intervals that they're not regulated by dim reg? Well, you could look at the soft integral and you could just do the perp. The perp is only showing up in this K squared minus M squared. So you would get something by doing that. And so, if we do the perp with dim reg and IS, we would end up proportional to something that's DK plus DK minus K plus K minus, minus M squared to the some power of epsilon, divided still by the factors of K plus and K minus.

So you see that the invariant mass is being regulated. We just did the perp. Perp is gone. Plus times minus is being regulated, because plus times minus-- if plus times minus grows larger, or goes small, and regulated by this epsilon-- but either one, plus going large or minus going large, or plus going small minus going small, with plus and times minus fixed is not regulated. And that's the rapidity divergence. If the invariant mass is fixed and K plus over K minus goes large or small.

So let me write it as K minus over K plus going to 0, or going to infinity. Let me say, with K plus times K minus fixed, then it diverges as these things happen. And if you think about what's happening in our picture over there within these limits, it's exactly a situation where this X here would be sliding up or sliding down.

So in one of these limits, this one's going towards the CN bar, and this one would be going towards CN. So it's exactly a region where you would be overlapping-- sliding down the hyperbola. And the interval has log singularities. So this is exactly a situation where we can't ignore the overlaps and we have to worry about them.

OK so we need another regulator. Dim reg is not enough. We have to do something else. So what could we do? So there's lots of different things that you could do. One thing is, you could just sort of put it in some plus something in these denominators-- that's called the delta regulated, K plus, plus delta-- that's one choice. We'll do something a little bit more dim reg-like. Which makes it sort of easier to think about the renormalization group. So one choice for an additional regulator is the following.

So if you think about where these divergences came from, they came from the Wilson lines. So what you really need to do is regulate the Wilson lines. And you can do that as follows. Let me write out the Wilson lines in our kind of momentum space notation. So we have some N dot P type momentum for the soft Wilson line, and an N dot AS field. And really, it's this one over this iconal denominator that's giving rise to these denominators here that are giving rise to the singularity.

So if we want to regulate that singularity we need to add something, and we could do that as follows. So this is the regulator we'll pick. And I'll just write everything as kind of a momentum operator. So I've just tucked the Z momentum in and raised it to some power. So PZ is the difference between P minus and P plus. And that seems kind of arbitrary, but that'll do the job for us.

You can motivate why you want to do PZ-- so this is 2PZ actually. You can motivate why you want to do PZ rather than something else in the following way. And it's a true fact, that once you have enough experience you realize it's good to use PZ, because PZ doesn't involve P0. And the softs don't really make a distinction between any of the different components. And if you put in P0, something that involved P0, that would be dangerous.

So this is nice, because there is no P0. So it's the combination of P plus and P minus that you can form that doesn't have the P0, which is energy. And remember that the polls in P0 are related to things like quarks and anti-quarks. They're related to unitarity. So not messing up the structure in P0 means that you'll be fine with unitarity, fine with causality, you're not messing up a lot of nice things about the theory.

So if you do put P0 in, then you have to be careful about those things. So if you just arbitrarily put in some power of P0, then you'd have more trouble. And so that's kind of why we're avoiding and just putting in PZ. For the colinears, we can do something similar. But for the colinears we also have a power counting between the minus and the plus.

So for the colinear, we can still make the power counting OK by thinking about putting in PZ, but then just expanding it to be a P minus. And that's true up to power corrections. And we don't really need to worry about power corrections when we're regulating these divergences. So it's just putting in the large momentum. And so W written in a similar notation. So I'll explain what the other things in this formula are in a minute. But the important thing for regulating is that we have some factor. In this case, it would be a factor of N bar dot P.

STUDENT: Dot [INAUDIBLE]?

PROFESSOR: Sorry?

STUDENT: Dot eta to bar dot P?

PROFESSOR: No. It's supposed to be an N. Looks like eta. Too many variables. There's the eta. So there's some factor raising of the-- again the iconal propagator sort of mixing up with the iconal propagator. In this case, it's even more obvious that it's just regulating that iconal propagator. OK.

So if we were to do that, and go back over here, and put the regulators into these integrals, what would happen? So here, we'd get an extra factor-- K minus to the eta. And so, that would regulate this K minus. And these integrals will also have polls from the 1 over K minus. You could think about-- well, OK. If we did those integrals, we would also have rapidity divergences that are kind of the analog ones, and the colinear SECT are still the soft ones.

Here we have two propagators. And so if I have two soft Wilson lines, and so I get two factors. But I've conveniently chose it to be the square root. So it comes out kind of looking the same here. So one thing that is just a part of this regulator-- which actually I don't know a good argument for-- is kind of a priority from the symmetries of the theory, you might like to argue that that should be eta over 2, and this should be eta. But it's really just part of-- it's just a choice, a convention that we've made, as far as I know. There probably is some nice deep argument for it, but I don't know it.

So what are these other factors? So nu is going to play the role of mu. We've changed the dimension of the operator. We've compensated it back with nu, just like we were doing with mu. We're going to get 1 over eta divergences, which are like our 1 over epsilon divergences. And we're going to get logs of nu, which are the analogs of logs of mu. And that's the sense in which there's kind of an analog of this M up with our usual dim reg setup.

In order to have a full analog, we should think about having a coupling. And so, that's what this W factor is. You can think about it like there was some bare pseudo coupling, which is really just 1. But just imagine that you're switching from bare to renormalized, in order to set up a renormalization group equation.

And then this guy here, which is in eta dimensions, if you like, would have a renormalization group which would say nu D by D nu of this W of eta and nu is minus eta over 2. Sorry, this is eta over 2. So that's the analog of saying mu by D mu of alpha is minus 2 epsilon alpha. So an analog statement. I think this is OK.

So this guy here is like a dummy coupling. And the boundary condition for it after you've carried out these-- this is just to set it to back to 1. So it's identically 1, it's really just a bookkeeping device. It's just--e it's a dummy coupling once you go to the eta dimensions. But you just set it always the renormalized coupling is just identically set to 1.

And identically setting it to 1 is what you need to keep gauge invariance in these Wilson lines. It turns out actually that this regulator here is gauge invariant, though it doesn't look like it. We've modified the structure of the Wilson line in some kind of way that looks like it might be drastic. But actually these factors here are gauge-- still leave a gauge invariant object. So--

STUDENT: Can you write [INAUDIBLE] space, I assume?

PROFESSOR: Not that I know of. Yeah.

STUDENT: I think [INAUDIBLE].

PROFESSOR: Maybe you can. Yeah. But it's not-- since it's not-- yeah, I don't know how to-- I don't know what it would look like. You could probably transform that power, and it--

STUDENT: [INAUDIBLE].

PROFESSOR: Yeah. I'm sure you can probably just try out before you [? transplant ?] that. I'm just not sure if it would look nice. Yeah. It might not look too bad. Yeah, and it might actually be a nice way of saying what I'm about to say in a less nice way, which is, if you look at the gauge symmetry, why is this not messing it up? So one way of thinking about that is just to look at general covariant gauge. So note, the 1 over eta and eta 0 terms are gauge invariant.

And you can think about that by just going to a general covariant gauge and seeing the parameter dependence drop out. So for example, at 1 loop, you would take [? g mu ?] nu in the contractions and replace it in general covariant gauge by some gauge parameter-- of general covariant gauge, K mu, K nu over K squared.

And you'd like to see it independent of this. But this eta to the 0 piece is kind of independent of that for the usual reasons. And the 1 over eta term is independent of that, because this guy actually doesn't deuce any rapidity singularities. What happens is that if you have an N dot K, then you have a corresponding N mu in the numerator. And so, basically what happens is, you get an extra N dot K in the numerator. So any time you have 1 over N dot K, you would get for this piece multiplied by an N dot K upstairs.

And so this is cancelling, you don't have a rapidity divergence in the C-dependent part. So that's why this is invariant under the gauge symmetry. And then, because of this boundary condition, the kind of cancellation of the C-dependence in the order [? A ?] to the 0 piece, this kind of works out in the standard way. So it gives you an idea of why it's gauge invariant without giving you a kind of full proof or anything.

So we have both 1 over epsilon polls and 1 over eta polls in general, and we have to understand what to do with them. So here's what we're going to do. For any fixed invariant mass, it turns out that we can have these one over eta polls. And the right procedure for dealing with them is as follows.

First you take eta goes to 0 and deal with these new polls that you have introduced in your amplitude. In order to deal with them, because you can have them for any invariant mass, you actually have to add counter terms that can be a whole function of epsilon, where you have an expanded in epsilon and then divide it by eta. So let me abbreviate counterterm as CT dot. Then, after you've done that, you take epsilon goes to 0, and you find your 1 over epsilon counterterms. And this is the correct way of doing it. And we'll see how that works in practice in a minute.

So let's go back to our integrals that I've now erased and just write out the answers. We're doing those integrals with this regulator. And I'll also make them fermions, so I'm putting in the numerators. We wrote them down for scalars. The scalars where the most divergent integrals actually. I can include the numerators, that doesn't really change the story. And I can include the pre-factors as well.

And I'll kind of write things in a QCD type notation, even we can imagine that it's a non-abelian group, just so CF is the whatever group it is, it's the Casimir of the fundamental. Whatever group our gauge boson's in. So here's the eta poll. It has a whole function of epsilon in the numerator, and it's even divergent. So this is 2 eta. And then the rest of it I can expand.

So there's going to be 1 over epsilon times the log. When the log replaces that 1 over epsilon then I can start to expand, and I get another 1 over-- when the log nu replaces the one over eta, I can expand this gamma, and it gives me 1 over epsilon. And there's also some other pieces. So over 2 epsilon there's a log mu over M. And there's a constant. And I'm never going to write the constants.

So let me read all the results and then we'll talk about them. So ICN bar is the same. The only difference between this is that P minus close to P bar plus. It was really symmetric. And then IS is different.

So the 1 over epsilon-- 1 over eta poll comes with the opposite sign, and it also comes to a factor of 2 different. And in this case, there's actually 1 over 2 epsilons squared term. So there's also a double log of mu or M. And then there's plus constant. OK?

And so, you could think about adding them up. And what happens when you add them up is, you have 1 over 2 eta, 1 over 2 eta, minus 1 over eta, and so that 1 over eta polls cancel. And that's exactly what you'd expect, because in the full theory the eta was something we introduced in order to distinguish these effective theory modes. It wasn't something that was there needed for the full theory integral. And so you don't really-- you'd expect that it's sort of-- that there's a corresponding regulator between the two sectors. So that when you add them together, that the dependence on that parameter is canceling away, because it was just an artificial separation, if you like, or separation that we're doing.

So if I add them up, 1 over eta is cancelled, and so do all the logs of nu. We have alpha-- I'm left with a log of mu over Q, 1 over epsilon poll, double log, some types of single logs, and some other type of double log. So it would look like that. All the nu dependence is canceling away. So sort of various things which we'll start talking about now, and we'll continue talking about next time.

So the rapidity 1 over eta divergence, which we can call a rapidity divergence, cancels in sum. And of course, so does the log nu's. And that's as expected.

And if you add an overall counterterm, for the entire thing it just involves the hard scale log mu over Q. So if you were to think about there being some Wilson coefficient, which is sort of C bare is ZC minus 1, Z bare is ZC, C renormalized, then ZC and the C renormalized only involve logs of mu over Q, which is the hard scale. OK?

And that means that our hard function, which is the Wilson coefficient squared, or just the Wilson coefficient in this case, is only a function of Q and mu. OK, so integrating out the hard scale physics didn't know about the separation. The separation was really something that we needed to do in the effective theory to distinguish the CN and S modes.

And you can see why we needed to do it if you look at these answers, because if you look at the types of logs that are showing up here, in this case, we have a nu over P minus. And in this case, we have a-- is it nu over mu? Just make sure I got that right. I guess it is. In this case, we have a nu over M. And we also have a mu over nu.

And so, the sort of right scale to-- in order to minimize the logarithms here we're going to have to, again, as usual take different values of mu and nu. Well, it's the same value of mu. All of them are M. But it's a different value of nu, because it's the nu that would need to be of order Q here. P minus is Q. And the nu would need to be of order M here. So it's the nu that distinguishes the modes. So the logs NCN are minimized. Or mu of order M, which says being on the hyperbola, but the nu should be of order P minus, which is Q.

And that's precisely actually where we put the X in our picture, if you think about it. OK, so that's saying that you have a large P minus momentum, and we have-- and we're on this hyperbola where P squared is an order M squared. So this is-- and it's likewise for the other pieces. So for the soft piece-- so for the-- say for the anti- for the other colinear piece, we need the same thing. And then for the soft we need a different value for this new parameter.

So having this regulator is behaving like dim reg, where we needed different mu's, when we had different hyperbolas. Now we have different places on the hyperbola, and we're tracking that with the new parameter, and that's showing up in the logarithms. And if you think about what the logarithms are doing, you can see that when you combine terms, let's see-- if you look at the 1 over epsilon, and the mu's are canceling out, you're getting a mu over Q. Yeah, that's maybe not the best example.

Look over here. You have this log of M over Q In this kind of complete decomposition. The way that that logarithm here gets made up is by having m over nu and nu over Q. All right? So in order to get this log that doesn't have any mu's in it, mu is not telling you that there's that large log. But there is a large log. So there's large logs associated to these rapidity divergences.

And what we'll talk about next time is how you do the renormalization group with the diagrams like this. How you write down in almost dimension equations. There'll be an almost dimension equations in both mu and nu space. So we'll have to move around in that space and see how it works to sum of the large logarithms. But I'll postpone that to next time. So any questions?

So the general idea is really, as usual, it's just that now we're dealing with a situation where there's two regulators. And they're actually independent regulators. One's, if you like, is regulating invariant mass, and the other is regulating these extra divergences. And so, we'll be able to move around in the space without any worrying about path dependence, for example. We'll talk a little bit about that next time-- in this two dimensional space of mu and nu.

But you can see just from looking at the logs, as usual, you can see where you need to be. And you can see that you need to be in different places for the different modes in order to minimize the logs of these amplitudes. And if you do that, then there should be some renormalization group that would connect these guys. So there should do something RGE that goes between these guys, and it'll be an RGE in this new parameter.