spectrum of the galaxy. Galaxies are a combination of these, so the total flux at a given frequency is a combination of the flux from each star:

$$F_\nu = N_O F_{\nu,O} + N_B F_{\nu,B} + ... \tag{27}$$

Determining $N_O, N_B, N_A, N_F...$ is the basic idea of stellar population synthesis.

2 Structure and a qualitative picture of galaxies

Goal: look at the most basic dynamical properties of a galaxy.

A *galaxy* is a collisionless fluid of stars and dark matter orbiting together with collisional gas in a common self-gravitational potential.

With this definition, we can try to understand the main dynamical properties.

2.A Virial Theorem

Derivation: assume stars orbit in a galaxy with mass, position, and velocity $(m_i, \vec{r}_i, \vec{v}_i)$. We then define the virial G:

$$G = \sum_i \vec{p}_i \cdot \vec{r}_i \tag{28}$$

which we can rewrite:

$$G = \sum_i \left(m_i \frac{d\vec{r}_i}{dt} \right) \cdot \vec{r}_i \,. \tag{29}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}(\vec{r} \cdot \vec{r}) = \dot{\vec{r}} \cdot \vec{r} + \vec{r} \cdot \dot{\vec{r}} = 2\dot{\vec{r}}\, \vec{r} \,, \tag{30}$$

we get

$$\begin{aligned} G &= \frac{1}{2} \sum_i m_i \frac{\mathrm{d}}{\mathrm{d}t} (\vec{r}_i \cdot \vec{r}_i) \\ &= \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \sum_i m_i r_i^2 \,. \end{aligned} \tag{31}$$

Defining $I = \sum_i m_i r_i^2$ as the moment of inertia about the origin, we get

$$\boxed{G = \frac{1}{2} \frac{\mathrm{d}I}{\mathrm{d}t}} \,. \tag{32}$$

Now consider the time derivative of G:

$$\begin{aligned} \frac{\mathrm{d}G}{\mathrm{d}t} &= \sum_i \dot{\vec{p}}_i \cdot \vec{r}_i + \sum_i \vec{p} \cdot \dot{\vec{r}}_i \\ &= \sum_i \vec{F}_i \cdot \vec{r}_i + \sum_i m_i v_i^2 \\ &= \sum_i \vec{F}_i \cdot \vec{r}_i + 2T \end{aligned} \tag{33}$$

where T is the kinetic energy. Because gravity is a pairwise force, we can write

$$\vec{F}_k = \sum_{i=1}^{N} \vec{F}_{jk} \ . \tag{34}$$

$F_{ii} = 0$ and $1 \leq j \leq N$, so we can split F_{jk} into two parts, the upper and lower portions of the matrix

$$\begin{array}{c} k \downarrow \ \ j \rightarrow \\ F_{jk} = \begin{pmatrix} 0 & & 2 \\ & \ddots & \\ 1 & & 0 \end{pmatrix} \end{array} \tag{35}$$

with

$$1 = \sum_{k=2}^{N} \sum_{j=1}^{k-1} \vec{F}_{jk} \cdot \vec{r}_k \qquad \text{and} \qquad 2 = \sum_{k=1}^{N-1} \sum_{j=k+1}^{N} \vec{F}_{jk} \cdot \vec{r}_k \tag{36}$$

so

$$\sum_{k=1}^{N} \vec{F}_k \cdot \vec{r}_k = \sum_{k=2}^{N} \sum_{j=1}^{k-1} \vec{F}_{jk} \cdot \vec{r}_k + \sum_{k=1}^{N-1} \sum_{j=k+1}^{N} \vec{F}_{jk} \cdot \vec{r}_k \ . \tag{37}$$

F is pairwise, so $-\vec{F}_{kj} = \vec{F}_{jk}$, which gives

$$\sum_{k=1}^{N} \vec{F}_k \cdot \vec{r}_k = \sum_{k=2}^{N} \sum_{j=1}^{k-1} \vec{F}_{jk} \cdot \vec{r}_k \ - \sum_{k=1}^{N-1} \sum_{j=k+1}^{N} \vec{F}_{kj} \cdot \vec{r}_k \ . \tag{38}$$

The second term in the above equation can be rewritten:

$$\sum_{k=1}^{N-1} \sum_{j=k+1}^{N} \vec{F}_{kj} \cdot \vec{r}_k = \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} \vec{F}_{jk} \cdot \vec{r}_j = \sum_{k=2}^{N} \sum_{j=1}^{k-1} \vec{F}_{jk} \cdot \vec{r}_j \tag{39}$$

which has the same matrix elements as the first term, so we get

$$\sum_{k=1}^{N} \vec{F}_k \cdot \vec{r}_k = \sum_{k=2}^{N} \sum_{j=1}^{k-1} \vec{F}_{jk} \cdot (\vec{r}_k - \vec{r}_j) \ . \tag{40}$$

We now assume that there is a potential V such that:

$$\begin{aligned} \vec{F}_{jk} &= -\nabla_k V\left(|\vec{r}_{jk}|\right) = -\nabla_k V(r_{jk}) \\ &= -\frac{\mathrm{d}V}{\mathrm{d}r} \left(\frac{\vec{r}_k - \vec{r}_j}{r_{jk}} \right) \end{aligned} \tag{41}$$

so we get

$$\begin{aligned}\sum_{k=1}^{N} \vec{F}_k \cdot \vec{r}_k &= \sum_{k=2}^{N}\sum_{j=1}^{k-1} \vec{F}_{jk} \cdot (\vec{r}_k - \vec{r}_j) \\ &= -\sum_{k=2}^{N}\sum_{j=1}^{k-1} \frac{\mathrm{d}V}{\mathrm{d}r}\frac{|\vec{r}_k - \vec{r}_j|^2}{r_{jk}} \\ &= -\sum_{k=2}^{N}\sum_{j=1}^{k-1} \frac{\mathrm{d}V}{\mathrm{d}r} r_{jk}\ .\end{aligned} \tag{42}$$

We now assume the special case $V(r_{jk}) = \alpha r_{jk}^n$. This gives us

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}r} &= n\alpha r_{jk}^{n-1} \\ \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}r} r_{jk} &= nV\end{aligned} \tag{43}$$

so

$$\begin{aligned}\sum_{k=1}^{N} \vec{F}_k \cdot \vec{r}_k &= -\sum_{k=2}^{N}\sum_{j=1}^{k-1} nV(r_{jk}) \\ &= -n\sum_{k-2}^{N}\sum_{j=1}^{k-1} V(r_{jk}) \\ &= -nV_{\text{tot}}\ .\end{aligned} \tag{44}$$

Finally:

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \sum_i \vec{I}_i \cdot \vec{r}_i + 2T = 2T - nV_{\text{tot}} \tag{45}$$

With $\frac{dG}{dt} = \frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2}, U = V_{\text{tot}}$, and $n = -1$ (for gravity):

$$\boxed{\frac{1}{2}\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 2T + U}\ . \tag{46}$$

We now take the time average:

$$\begin{aligned}\left\langle \frac{\mathrm{d}G}{\mathrm{d}t} \right\rangle_{\mathcal{T}} &= \frac{1}{\mathcal{T}}\int_0^{\mathcal{T}} \frac{\mathrm{d}G}{\mathrm{d}t}\mathrm{d}t = \frac{G(\mathcal{T}) - G(0)}{\mathcal{T}} \\ \Rightarrow \left\langle \frac{\mathrm{d}G}{\mathrm{d}t} \right\rangle_{\mathcal{T}} &= 2\langle T\rangle_{\mathcal{T}} - n\langle V_{\text{tot}}\rangle_{\mathcal{T}}\end{aligned} \tag{47}$$

For a steady state system and long time average, $\frac{G(\mathcal{T})-G(0)}{\mathcal{T}} \approx 0$, so we get

$$\boxed{\begin{aligned}&0 = 2\langle T\rangle_{\mathcal{T}} - n\langle V_{\text{tot}}\rangle_{\mathcal{T}} \\ &0 = 2\langle T\rangle_{\mathcal{T}} + \langle U\rangle_{\mathcal{T}} \qquad \text{for } n = -1\end{aligned}}\ . \tag{48}$$

This is the *virial theorem*, often written simply as $0 = 2T + U$.

Note the three important assumptions for the virial theorem to hold:

- F is a pairwise force
- The potential V has the form $V \propto r^n$
- We have a steady state time averaged quantity $\frac{d^2 I}{dt^2} = 0$.

Applications:
We now apply the virial theorem to a galaxy.

$$2T + U = 0 \ . \tag{49}$$

Assuming that a galaxy is made of N stars all with the same mass m (so total mass $M = Nm$) and average velocity $\vec{v}$, we get a total kinetic energy for the system

$$T = \frac{1}{2}\sum_i m_i v_i^2 \approx \frac{1}{2} M\bar{v}^2 = \frac{1}{2} M v^2 \ . \tag{50}$$

From dimensional analysis for a galaxy of size R, we get a total potential energy

$$U = -\frac{GM^2}{R} \ . \tag{51}$$

The virial theorem then implies

$$\begin{aligned} Mv^2 + \left(-\frac{GM^2}{R}\right) &= 0 \\ \Rightarrow v &= \sqrt{\frac{GM}{R}} \ . \end{aligned} \tag{52}$$

Using some typical numbers:
$R \approx 10\text{kpc}$ $\quad M_\odot = 2 \times 10^{33}\text{g}$
$M \approx 10^{11} M_\odot$ $\quad G = 0.0043 M_\odot^{-1}\text{pc}\left(\frac{\text{km}}{\text{s}}\right)^2$

$$\begin{aligned} v &= \sqrt{\frac{0.0043 M_\odot^{-1}\text{pc}(\text{km/s})^2 \ 10^{11} M_\odot}{10\,000\text{pc}}} \\ &\approx \sqrt{4 \times 10^4}\ \text{km/s} \approx 200\ \text{km/s} \end{aligned} \tag{53}$$

which is in good agreement with observations.

We can also use the virial to get the ideal gas law.
For an ideal gas with N particles at temperature T is

$$K = \frac{3}{2} NkT \tag{54}$$

where we use K for kinetic energy to differentiate from temperature and k is the Boltzmann constant. The force comes from the pressure from the particles, so the force per unit area is

$$\mathrm{d}\vec{F} = -P\mathrm{d}\vec{A} \ . \tag{55}$$

Then the potential energy is

$$-\frac{1}{2}\left\langle \sum_i \vec{F}_i \cdot \vec{r} \right\rangle = \frac{P}{2}\int \vec{r}_i \cdot \mathrm{d}\vec{A} \tag{56}$$

and $n = 2$. From the divergence theorem

$$\begin{aligned} \int \vec{r}_i \cdot \mathrm{d}\vec{A} &= \int \vec{\nabla} \cdot \vec{r} \mathrm{d}V \\ &= 3\int \mathrm{d}V = 3V \ . \end{aligned} \tag{57}$$

The virial theorem gives

$$\begin{aligned} 0 &= 2\left\langle K \right\rangle_{\mathcal{T}} - 2\left\langle V_{\mathrm{tot}} \right\rangle \\ &= 2\frac{3}{2}NkT - 2\frac{P}{2}3V \end{aligned} \tag{58}$$

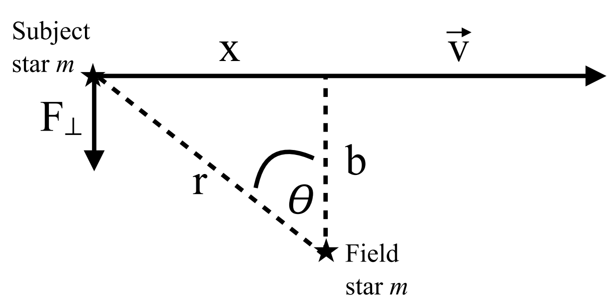
so

$$NkT = PV \ . \tag{59}$$

2.B Relaxation times

The virial theorem gave us some first insight into the dynamics of galaxies. Now we will show that stars are collisionless, i.e. that two-body collisions are rare in galaxies. Since this is true, we can describe the distribution of stars as a smooth density field and gravitational potential.

Frequency of strong encounters between stars:
Goal: estimate the change in velocity $\delta\vec{v}$ by which the encounter deflects the velocity $\vec{v}$ of the subject star.



We assume that $|\delta\vec{v}|/|\vec{v}| \ll 1$ and that the field star is stationary. This means that $\delta\vec{v}$ is perpendicular to $\vec{v}$ since the accelerations parallel to $\vec{v}$ cancel out as the subject star passes by the field star. We calculate $\delta v = |\delta\vec{v}|$ by integrating $F_\perp$:

$$F_\perp = \frac{Gm^2}{b^2 + x^2}\cos\theta = \frac{Gm^2 b}{(b^2 + x^2)^{3/2}} = \frac{Gm^2}{b^2}\left[1 + \left(\frac{vt}{b}\right)^2\right]^{-3/2} \ . \tag{60}$$