Then the potential energy is

$$-\frac{1}{2}\left\langle \sum_i \vec{F}_i \cdot \vec{r} \right\rangle = \frac{P}{2}\int \vec{r}_i \cdot \mathrm{d}\vec{A} \tag{56}$$

and $n = 2$. From the divergence theorem

$$\begin{aligned} \int \vec{r}_i \cdot \mathrm{d}\vec{A} &= \int \vec{\nabla} \cdot \vec{r} \mathrm{d}V \\ &= 3\int \mathrm{d}V = 3V \ . \end{aligned} \tag{57}$$

The virial theorem gives

$$\begin{aligned} 0 &= 2\left\langle K \right\rangle_{\mathcal{T}} - 2\left\langle V_{\mathrm{tot}} \right\rangle \\ &= 2\frac{3}{2}NkT - 2\frac{P}{2}3V \end{aligned} \tag{58}$$

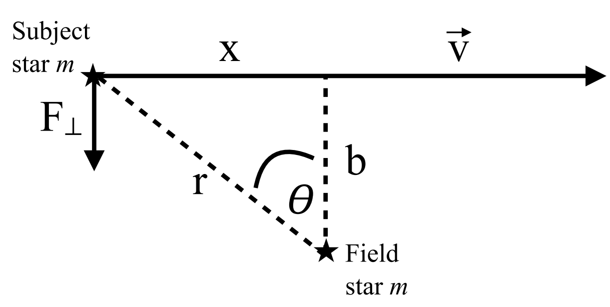
so

$$NkT = PV \ . \tag{59}$$

2.B Relaxation times

The virial theorem gave us some first insight into the dynamics of galaxies. Now we will show that stars are collisionless, i.e. that two-body collisions are rare in galaxies. Since this is true, we can describe the distribution of stars as a smooth density field and gravitational potential.

Frequency of strong encounters between stars:
Goal: estimate the change in velocity $\delta\vec{v}$ by which the encounter deflects the velocity $\vec{v}$ of the subject star.



We assume that $|\delta\vec{v}|/|\vec{v}| \ll 1$ and that the field star is stationary. This means that $\delta\vec{v}$ is perpendicular to $\vec{v}$ since the accelerations parallel to $\vec{v}$ cancel out as the subject star passes by the field star. We calculate $\delta v = |\delta\vec{v}|$ by integrating $F_\perp$:

$$F_\perp = \frac{Gm^2}{b^2 + x^2}\cos\theta = \frac{Gm^2 b}{(b^2 + x^2)^{3/2}} = \frac{Gm^2}{b^2}\left[1 + \left(\frac{vt}{b}\right)^2\right]^{-3/2} \ . \tag{60}$$

Newton's law $m\dot{\vec{v}} = \vec{F}$ gives us the change in the perpendicular velocity

$$\begin{aligned}\delta v &= \frac{1}{m}\int_{-\infty}^{+\infty} \mathrm{d}t F_{\perp} \\ &= \frac{Gm}{b^2}\int_{-\infty}^{+\infty} \frac{\mathrm{d}t}{\left[1+\left(\frac{vt}{b}\right)^2\right]^{3/2}} \\ &= \frac{Gm}{bv}\int_{-\infty}^{+\infty}\frac{\mathrm{d}s}{(1+s^2)^{3/2}} \\ &= \frac{2Gm}{bv}\end{aligned} \tag{61}$$

using $s = \frac{vt}{b}$. Thus, δv is roughly equal to the acceleration at closest approach, $\frac{GM}{b^2}$, times the duration of the acceleration, $\frac{2b}{v}$.

Strong encounters:
An encounter is strong if $\delta v \sim v$ (which also causes the calculation to break down). This is also when a star will have its path deflected by $\sim 90° \equiv b_{\text{strong}}$.

$$\delta v \sim v \Leftrightarrow b \lesssim b_{90} = \frac{GM}{v^2} \equiv b_{\text{strong}} \ . \tag{62}$$

The cross section for strong encounters is

$$\sigma_{\text{strong}} = \pi b_{\text{strong}}^2 \tag{63}$$

From the virial theorem, we have

$$\begin{aligned} v^2 &\sim \frac{GM}{R} = \frac{GNm}{R} \\ \Rightarrow b_{\text{strong}} &\approx \frac{2R}{N_*}\end{aligned} \tag{64}$$

so we get:

$$\sigma_{\text{strong}} \approx \frac{4\pi}{N_*^2}R^2 \tag{65}$$

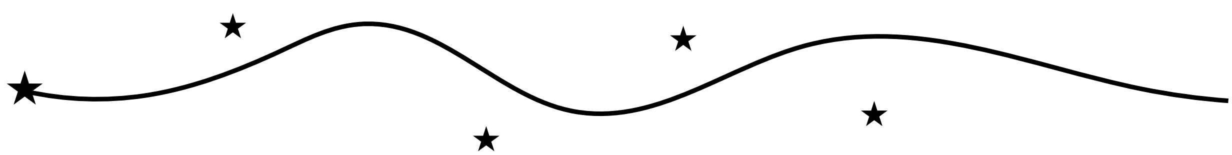
which is small since $N \sim 10^{11}$. This means that the probability p of a strong encounter over a single crossing of a star through a galaxy with an average number density of stars n is

$$\begin{aligned} p &= n\sigma_{\text{strong}}R \\ &= \frac{N}{\frac{4}{3}\pi R^3}\frac{4\pi R^2}{N^2}R \\ &= \frac{3}{N} \sim 10^{-11} \ . \end{aligned} \tag{66}$$

This is a tiny probability! So there are likely no strong encounters in a galaxy. For globular clusters, $N \sim 10^4$, so strong encounters are more common.

What about weak encounters?
We have seen that strong encounters are rare, i.e. they practically never happen. Nevertheless, if a star crosses a galaxy many times, it will encounter many weak encounters. Each of those will slightly perturb its velocity until $v_\perp \approx v$. The time it takes for this to happen is the relaxation time of the system.

Multiple weak encounters

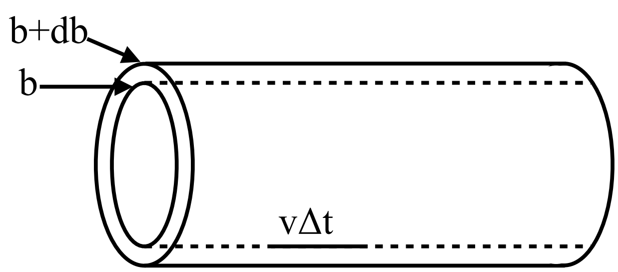


A star makes a random walk through a galaxy. Its total deviation from its path is the sum of each of its encounters with other stars. For N encounters,

$$\delta v_{\rm tot}^2 = \sum_{i=1}^{N} (\delta v_i)^2 \, . \tag{67}$$

The strength of each encounter depends on the impact parameter b. The number of encounters N within $(b, b+db)$ is

$$N = \underbrace{(2\pi b db)}_{\text{area}} \; \underbrace{(v\Delta t)}_{\text{length}} \; \underbrace{n}_{\text{density}} \, . \tag{68}$$



We then have

$$\begin{aligned}
\sum_i (\delta v_i)^2 &= \int_{b_{\rm min}}^{b_{\rm max}} (\text{number of encounters in } (b, b+\mathrm{d}b)) \times (\delta v \text{ for each encounter with } b) \\
&= \int_{b_{\rm min}}^{b_{\rm max}} (2\pi \, v \, \Delta t \, n \, b \, \mathrm{d}b) \left(\frac{2Gm}{bv}\right)^2 \\
&= \frac{8\pi G^2 m^2 n}{v} \Delta t \int_{b_{\rm min}}^{b_{\rm max}} \frac{\mathrm{d}b}{b} \, .
\end{aligned} \tag{69}$$

Now we determine the limits on the integral:

$$\begin{aligned} b_{\max} &\approx R \approx 10\,\mathrm{kpc} \\ b_{\min} &\approx b_{\mathrm{strong}} = \frac{2R}{v_*} = 10^{-10}\mathrm{kpc}\ (0.01\mathrm{AU}) \end{aligned} \tag{70}$$

so

$$\begin{aligned} \int_{b_{\min}}^{b_{\max}} \frac{\mathrm{d}b}{b} &= \ln\left(\frac{b_{\max}}{b_{\min}}\right) \equiv \ln\Lambda \quad \text{(Coulomb logarithm)} \\ &= \ln\left(\frac{10\ \mathrm{kpc}}{10^{-10}\ \mathrm{kpc}}\right) \\ &= \ln\left(10^{11}\right) \approx 25\ . \end{aligned} \tag{71}$$

Relaxation time:
We define the relaxation time t_{relax} through

$$\begin{aligned} \sum_i (\delta v_i)^2 \approx v^2 \Rightarrow \frac{8\pi G^2 m^2 n}{v} \ln\Lambda t_{\mathrm{relax}} \\ \Rightarrow t_{\mathrm{relax}} = \frac{v^3}{8\pi G^2 m^2 n \ln\Lambda}\ . \end{aligned} \tag{72}$$

We now compare this to the dynamical time $t_{\mathrm{orbit}} \approx R/v$ of the system:

$$\frac{t_{\mathrm{relax}}}{t_{\mathrm{orbit}}} = t_{\mathrm{relax}} \frac{v}{R} = \frac{v^4}{8\pi G^2 m^2 n \ln\Lambda R}\ ,. \tag{73}$$

Using the virial theorem $v^2 = \frac{GM}{R}$ and number density $n = \frac{M/m}{\frac{4\pi}{3}R^3}$ gives us:

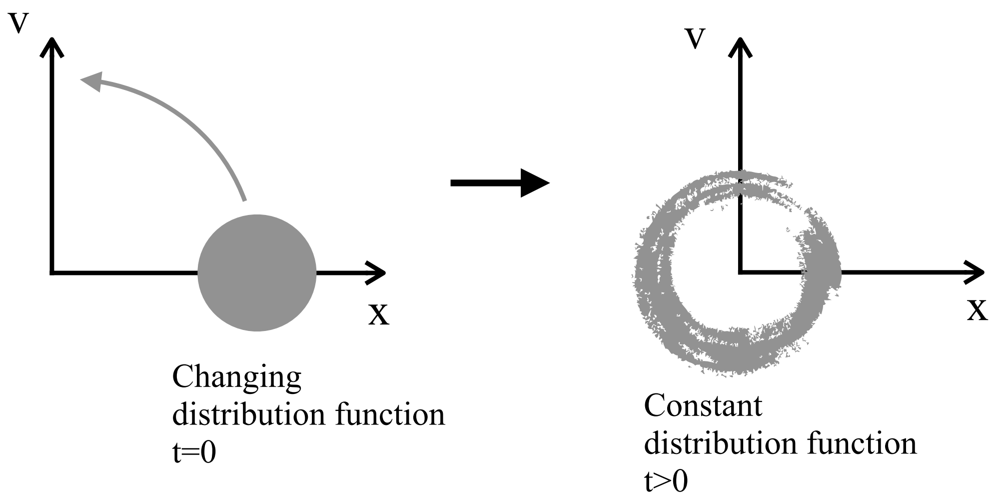
$$\begin{aligned} &= \frac{(GM/R)^2}{8\pi G^2 m^2 \frac{M/m}{\frac{4\pi}{3}R^3} R \ln\Lambda} \\ &= \frac{M}{8\pi m \frac{3}{4\pi} \ln\Lambda} \\ &= \frac{N}{6\ln\Lambda} = \frac{N_*}{6\ln\left(\frac{b_{\max}}{b_{\min}}\right)} = \frac{N_*}{6\ln\left(\frac{R}{2R/N_*}\right)} \\ &\sim \frac{N_*}{6\ln N_*} \end{aligned} \tag{74}$$

which is very large! Thus, stars are orbiting in an unperturbed collective potential (collisionless)!

2.C Collisionless relaxation

Relaxation occurs in two ways within a galaxy: the collisional gas with interactions reaches a Maxwellian distribution through two-body interactions, but the collisionless systems (stars and dark matter) must relax through a different process otherwise galaxies and galaxy clusters would not reach a relaxed state within the age of the Universe. We say a system is *relaxed* when its coarse grained phase-space distribution function does not change any more.

Collisionless relaxation processes:
Phase mixing:
The coarse grained phase-space distribution function is distributed over time so doesn't change with time.



Violent relaxation:
Since energy in the stellar and dark matter systems in galaxies can't be efficiently exchanged through collisions, we must find another way for energy exchange. The energy of an individual star (specific energy) is:

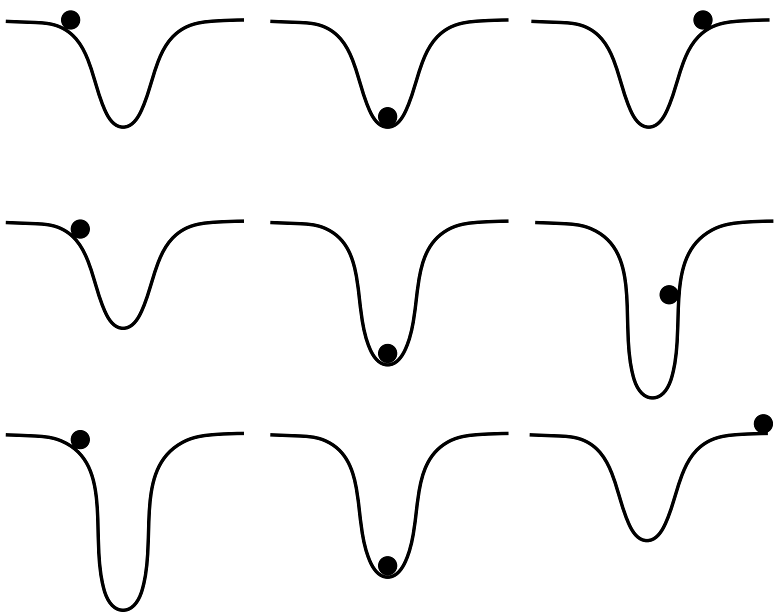
$$E = \frac{1}{2}v^2 + \phi \; . \tag{75}$$

Then the change in energy over time is

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\partial E}{\partial \vec{v}}\frac{\mathrm{d}\vec{v}}{dt} + \frac{\partial E}{\partial \phi}\frac{\mathrm{d}\phi}{\mathrm{d}t} \\ &= -\vec{v}\cdot\vec{\nabla}\phi + \frac{\mathrm{d}\phi}{\mathrm{d}t} \\ &= -\vec{v}\cdot\vec{\nabla}\phi + \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial\vec{x}}\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \\ &= -\vec{v}\cdot\vec{\nabla}\phi + \frac{\partial\phi}{\partial t} + \vec{v}\vec{\nabla}\phi \\ &= \frac{\partial\phi}{\partial t} \end{aligned} \tag{76}$$

Thus, the only way for a star to change its energy is by having a time-dependent potential.

To think of this intuitively, we can consider an object moving through a potential well. If the potential is constant with time, the particle will recover the same energy as it comes out the other side and there is no relaxation. If the potential grows with time, the particle will need to expend more energy to cross it and will not have enough energy to get back out of the potential well, thus losing energy. If the potential shrinks with time, the particle will gain energy as it crosses the well.



As a galaxy or cluster forms, the gravitational potential changes significantly as mass accretes and collapses into a halo. Averaging over all particles, the timescale for violent relaxation $t_{\rm vr}$ is

$$\begin{aligned} t_{\rm vr} &= \left\langle \frac{\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^2}{E^2} \right\rangle^{-1/2} \\ &= \left\langle \frac{\left(\frac{\partial \phi}{\partial t}\right)^2}{E^2} \right\rangle^{-1/2} \\ &\sim \left\langle \frac{\dot{\phi}^2}{\phi} \right\rangle^{-1/2} \end{aligned} \tag{77}$$

where in the last step we used the time-dependent virial theorem (see Lynden-Bell 1967). This occurs on roughly the same timescale as free-fall since this is the timescale at which the potential changes during collapse. It's very fast, hence 'violent' relaxation!

3 Modelling galaxies

So far, we have looked at the basic dynamical properties of galaxies. Now we discuss the main ingredients of modelling galaxies:

- potential-density pairs (the common potential)